

Basic Twist Quantization of the Exceptional Lie Algebra g_2

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We present the formulae for twist quantization of g_2 , corresponding to the solution of classical YB equation with support in the 8-dimensional Borel subalgebra of g_2 . The considered chain of twists consists of the four factors describing the four steps of quantization: Jordanian twist, the two twist factors extending Jordanian twist and the deformed Jordanian or in second variant additional Abelian twist. The first two steps describe as well the $sl(3)$ quantization. The coproducts are calculated for each step in explicit form, and for that purpose we present new formulas for the calculation of similarity transformations on tensor product. We introduce new basic generators in universal enveloping algebra $U(g_2)$ which provide nonlinearities in algebraic sector maximally simplifying the deformed coproducts.

I. INTRODUCTION

In this paper we shall consider the basic nonstandard quantum deformations of complex exceptional Lie algebra g_2 . There are four complex semisimple Lie algebras of rank 2, given by $A_2 \simeq sl(3)$, $D_2 \simeq o(4) = o(3) \oplus o(3)$, $B_2 \simeq C_2 \simeq o(5) \simeq sp(4)$ and g_2 , with 8, 6, 10 and 14 generators respectively. The 8-dimensional carrier of classical r -matrices which describe our deformations is equal to the Borel subalgebra $\mathfrak{b}_+(g_2)$ of g_2 .

There are two natural embeddings related with the group G_2 :

i) $G_2 \subset O(7)$. The fundamental matrix representation of G_2 is seven-dimensional. The 7×7 orthogonal matrices $(L_{ab}) \in O(7)$ ($a, b = 1, 2, \dots, 7$) belong to the group G_2 if the following cubic constraint is satisfied [1, 2]

$$f_{a_1 a_2 a_3} = f_{b_1 b_2 b_3} L_{a_1 b_1} L_{a_2 b_2} L_{a_3 b_3}, \quad (I.1)$$

where the totally antisymmetric cubic tensor f_{abc} describes the multiplication table for imaginary octonions t_a

$$t_a t_b = f_{abc} t_c \quad (I.2)$$

and the values of f_{abc} are determined by the following choice (we list only nonvanishing values)

$$f_{127} = f_{157} = f_{163} = f_{264} = f_{245} = f_{374} = f_{576} = 1. \quad (I.3)$$

Therefore there are only seven independent equations (I.1) reducing 21 parameter of $O(7)$ to 14 parameters of the group G_2 .

We see that the fundamental seven-dimensional representation $\{7\}$ of Lie algebra g_2 inherits basic properties of the fundamental $o(7)$ representation: reality and its dimensionality. The generators (E_k^l, A_k, B^l) ($k, l = 1, 2, 3$) of g_2 satisfy the following relations ([1]–[5])

$$[E_k^l, E_m^n] = \delta_m^l E_k^n - \delta_k^n E_m^l, \quad (I.4a)$$

$$[E_k^l, A_m] = \delta_m^l A_k - \frac{1}{3} \delta_k^l A_m, \quad (I.4b)$$

$$[E_k^l, B^n] = -\delta_k^n B^l + \frac{1}{3} \delta_k^l B^n, \quad (I.4c)$$

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$$[A_m, B^n] = E_m^n, \quad (\text{I.4d})$$

$$[A_m, A_n] = -\frac{4}{3} \epsilon_{mnl} B^l, \quad (\text{I.4e})$$

$$[B^m, B^n] = \epsilon^{mnl} A_l, \quad (\text{I.4f})$$

where

$$E_k^k = 0, \quad (\text{I.4g})$$

and we employ the $su(3)$ tensorial basis.

ii) $sl(3) \subset g_2$ or $su(3) \subset g_2$ (real form). The generators E_k^l forming the subalgebra (I.4a) describe $sl(3)$ (if g_2 is complex Lie algebra) or $su(3)$ (if we introduce in g_2 the suitable real structure). The Lie algebra generators of g_2 belong to 14-dimensional adjoint representation $\{\underline{14}\}$ which decomposes under $sl(3)$ (or $su(3)$) as follows:

$$\{\underline{14}\} = \{\underline{8}\} + \{\underline{3}\} + \{\bar{\underline{3}}\}. \quad (\text{I.5})$$

In the realization (I.4a–I.4f) of the Lie algebra g_2 the generators A_m (B^m) transform as fundamental triplet (antitriplet) representations of $sl(3)$ or $su(3)$. These properties can be also seen from the root diagram of g_2 (see Sect. II).

The relations (I.4d–I.4g) show that the coset space $S^6 = \frac{G_2}{SU(3)}$ is a nonsymmetric Riemannian space, with torsion described by nonvanishing rhs of relations (I.4e–I.4f). One of the aims of this paper is to provide an algebraic ground for new quantum deformation of the sphere S^6 with torsion.

The embedding of 3-dimensional fundamental representation of $su(3)$ described by Gell-Mann fundamental matrices λ_k ($k = 1, \dots, 8$) into 7-dimensional fundamental representation of g_2 looks as follows

$$\Lambda_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_k & 0 & 0 \\ 0 & -\lambda_k^* & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{I.6})$$

where we use the standard normalization

$$Tr \lambda_k \lambda_l = Tr \Lambda_k \Lambda_l = 2\delta_{kl}. \quad (\text{I.7})$$

Two Cartan generators of g_2 we identify with the $su(3)$ generators Λ_3 and Λ_8 .

The plan of our paper is the following:

In Sect. II we shall consider the Lie algebra g_2 in Cartan-Weyl basis (see e.g. [6]) which is directly linked with the generators (E_k^l, A_k, B^l) satisfying the algebra (I.4a–I.4g). We present the important class of triangular r -matrices for g_2 , satisfying the classical Yang-Baxter equation (CYBE). It appears that the two-parameter families of such r -matrices have as its carrier algebra the whole 8-dimensional Borel subalgebra $\mathfrak{b}_+(g_2) \subset g_2$. We show that the parameters of the considered classical r -matrices can achieve fixed nonzero values by means of inner automorphism maps inside g_2 algebra. In Sect. III we shall recall the general formulae which describe the twist quantization method [7]–[12], and we shall introduce the general twisting function, describing the twist quantization procedure for g_2 with the 8-dimensional carrier space for its r -matrix. In Sect. IV we obtain firstly the explicite formulae describing the twist quantization of g_2 generated by the $sl(3)$ classical r -matrix. It appears that these quantization formulae for g_2 describe the extension of known relations describing the twist quantization of $sl(3)$ [12]. In particular following general technique presented in [13] we shall introduce suitable nonlinear basis in the deformed Hopf algebras. In Sect. V we consider the most general g_2 -quantizations containing two additional twists, depending on the g_2 generators from the coset $\frac{G_2}{sl(3)}$. In Sect. VI we present a general discussion and some conclusions. We remark that in Sect. IV and V we shall use new algebraic formulae for calculating twisted coproducts presented in Sect III.D and shall introduce new basis of $U(g_2)$ which will simplify the twisted coproduct formulae.

The motivation for our work is mainly to present a new mathematical result - interesting class of quantum deformations for an important Lie algebra. On the other side it should be stressed that g_2 algebra recently has attracted attention of physicists in the domain of elementary particle physics and fundamental interactions theory. In particular we recall that:

i) In eleven-dimensional M -theory there were proposed the internal manifolds with g_2 holonomy as a base for the grand unification describing extension of the standard model in particle physics (see e.g. [14]–[17]). The algebra g_2 implies seven-dimensional internal symmetry space as the privileged one, in obvious connection with the relation $11 = 4 + 7$.

ii) In the reduction of supersymmetric theories from $D = 11$ to $D = 4$ the g_2 internal symmetry implies phenomenologically interesting case of $D = 4$ models with $N = 1$ supersymmetry [5]. In particular there were also considered standard and supersymmetric extensions of $D = 4$ chromodynamics to G_2 gauge theories [5] with interesting exceptional quark confinement mechanism.

iii) There are four Hurwitz algebras (real numbers R , complex numbers C , quaternions H and octonions O); G_2 acts on seven imaginary octonionic units and describes the automorphism group of the octonion algebra. All applications of exceptional and octonions groups to the description of symmetries in elementary particle physics (see e.g. [18]) is strongly linked therefore with the appearance of G_2 symmetry.

In this paper we consider only the quantum deformations of universal enveloping algebra $U(g_2)$; it is an interesting problem to supplement the considerations with deformations of dual Hopf algebra describing matrix quantum G_2 group and further describe e.g. the quantum deformations of $S^6 = \frac{G_2}{SL(3)}$. In such a way one can obtain an example of six-dimensional counterpart of two-dimensional Podleś sphere [19], provided by the deformed coset $\frac{SU(2)}{U(1)}$.

II. CARTAN-WEYL BASIS OF g_2 AND JORDANIAN TYPE CLASSICAL r -MATRICES

A. Cartan-Weyl Basis of g_2

In order to describe Cartan-Weyl basis of g_2 let us introduce the Dynkin diagram for its simple roots $\Pi = \{\alpha_1, \alpha_2\}$:

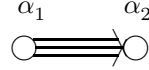


Fig. 1. Dynkin diagram of the Lie algebra g_2 .

The corresponding standard $A = (a_{ij})(i, j = 1, 2)$ and symmetric $A^{sym} = (a_{ij}^{sym})_{i,j}$ Cartan matrices are given by

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{sym} = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}. \quad (\text{II.1})$$

The Lie algebra g_2 is generated by the six Chevalley elements $e_{\alpha_i}, e_{-\alpha_i}, h_{\alpha_i}$ ($i = 1, 2$) with the defining relations (see e.g. [6])

$$\begin{aligned} [h_{\alpha_i}, h_{\alpha_j}] &= 0, \\ [h_{\alpha_i}, e_{\pm\alpha_j}] &= \pm a_{ij}^{sym} e_{\pm\alpha_j}, \\ [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij} h_{\alpha_i}, \\ [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_2}]] &= 0, \\ [[[[e_{\pm\alpha_1}, e_{\pm\alpha_2}], e_{\pm\alpha_2}], e_{\pm\alpha_2}], e_{\pm\alpha_2}] &= 0. \end{aligned} \quad (\text{II.2})$$

The positive $\Sigma_+(g_2)$ and total $\Sigma(g_2) = \Sigma_+(g_2) \cup (-\Sigma_+(g_2))$ root systems of g_2 are presented in terms of an orthonormalized basis $\{\epsilon_1, \epsilon_2\}$ of a 2-dimensional Euclidian space as follows

$$\Sigma_+(g_2) = \left\{ \sqrt{3}\epsilon_1, \epsilon_2, \frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{1}{2}\epsilon_2, \frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{3}{2}\epsilon_2 \right\} \quad (\text{II.3})$$

$$\Sigma(g_2) = \left\{ \pm\sqrt{3}\epsilon_1, \pm\epsilon_2, \pm\frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{1}{2}\epsilon_2, \pm\frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{3}{2}\epsilon_2 \right\}. \quad (\text{II.4})$$

where the simple roots are given by $\alpha_1 = \frac{\sqrt{3}}{2}\epsilon_1 - \frac{3}{2}\epsilon_2$ and $\alpha_2 = \epsilon_2$. It is convenient to present the total root system by the root diagram presented on Fig. 2.

For construction of the composite root vectors e_γ , ($\gamma \neq \pm\alpha_1, \pm\alpha_2$), we fix the following normal ordering of the positive root system $\Sigma_+(g_2)$ (see [6])

$$\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, \alpha_2, \quad (\text{II.5})$$

which corresponds to "clockwise" ordering for positive roots in Fig. 2 if we start from the root α_1 to the root α_2 . For convenience we introduce the short notations

$$e_{k,l} := e_{k\alpha_1 + l\alpha_2}, \quad h_{k,l} := kh_{\alpha_1} + lh_{\alpha_2} \quad (\text{II.6})$$

for $k, l = 0, \pm 1, \dots$. According to the ordering (II.5) we set the composite roots generators with suitably chosen numerical coefficients as follows

$$\begin{aligned} e_{1,1} &= [e_{1,0}, e_{0,1}], & e_{-1,-1} &= -[e_{-1,0}, e_{0,-1}], \\ e_{1,2} &= [e_{1,1}, e_{0,1}], & e_{-1,-2} &= -\frac{3}{4}[e_{0,-1}, e_{-1,-1}], \\ e_{1,3} &= [e_{1,2}, e_{0,1}], & e_{-1,-3} &= -\frac{3}{4}[e_{0,-1}, e_{-1,-2}], \\ e_{2,3} &= [e_{1,3}, e_{1,0}], & e_{-2,-3} &= -\frac{3}{4}[e_{-1,0}, e_{-1,-3}]. \end{aligned} \quad (\text{II.7})$$

The complete set of relations for Cartan-Weyl basis of g_2 can be read off from the formulae (I.4a–I.4f) after the identification

$$h_{1,0} \equiv h_1 = E_2^2 - E_3^3, \quad h_{0,1} \equiv h_2 = \frac{1}{6}(E_1^1 - 2E_2^2 + E_3^3) \quad (\text{II.8a})$$

and

$$\begin{aligned} e_{1,0} &= E_2^3, & e_{-1,0} &= E_3^2, \\ e_{0,1} &= B^2, & e_{0,-1} &= A_2, \\ e_{1,1} &= -B^3, & e_{-1,-1} &= -A_3, \\ e_{1,2} &= A_1, & e_{-1,-2} &= B^1, \\ e_{1,3} &= E_1^2, & e_{-1,-3} &= E_2^1, \\ e_{2,3} &= E_1^3, & e_{-2,-3} &= E_3^1. \end{aligned} \quad (\text{II.8b})$$

B. Jordanian Type Classical r -Matrices for g_2

Firstly we introduce some definitions concerning classical r -matrices. Let \mathfrak{g} be any simple Lie algebra then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where \mathfrak{n}_\pm are maximal nilpotent subalgebras and \mathfrak{h} is a Cartan subalgebra. The subalgebra \mathfrak{n}_+ (\mathfrak{n}_-) is generated by the positive (negative) root vectors e_β ($e_{-\beta}$) for all $\beta \in \Sigma_+(\mathfrak{g})$. The symbol \mathfrak{b}_+ will denote the Borel subalgebra of \mathfrak{g} , $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$. Let the elements $h_\theta \in \mathfrak{h}$ and $e_\theta \in \mathfrak{n}_+$ satisfy the relation

$$[h_\theta, e_\theta] = e_\theta. \quad (\text{II.9})$$

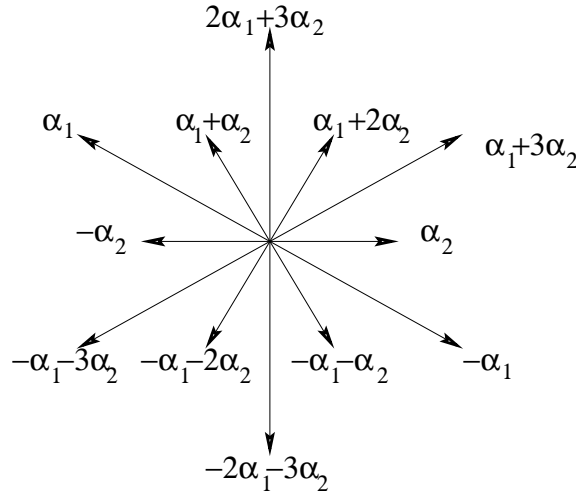


FIG. 2: The root diagram for g_2

A two-tensor $r_J(\xi_\theta) \in \mathfrak{b}_+ \otimes \mathfrak{b}_+$ of the form

$$r_\theta(\xi_\theta) = \xi_\theta h_\theta \wedge e_\theta := \xi_\theta (h_\theta \otimes e_\theta - e_\theta \otimes h_\theta) \quad (\text{II.10})$$

satisfies CYBE and it is called the Jordanian classical r -matrix. The symbol $\xi_\theta \in \mathbb{C}$ is a deformation parameter. Moreover, let elements $e_{\gamma_{\pm i}}$, indexed by the symbols i and $-i$, $i \in I = \{1, 2, \dots, N\}$ satisfy the relations

$$[h_\theta, e_{\gamma_i}] = (1 - t_{\gamma_i}) e_{\gamma_i}, \quad [h_\theta, e_{\gamma_{-i}}] = t_{\gamma_i} e_{\gamma_{-i}}, \quad (\text{II.11a})$$

$$[e_{\gamma_i}, e_{-\gamma_j}] = \delta_{ij} e_{\gamma_0} \quad [e_{\gamma_{\pm i}}, e_{\gamma_{\pm j}}] = 0, \quad (\text{II.11b})$$

$$[e_{\gamma_{\pm i}}, e_{\gamma_0}] = 0. \quad (\text{II.11c})$$

It is not difficult to check (see also [8]) that the element

$$r_{\theta;N}(\xi_\theta) = \xi_\theta \left(h_\theta \wedge e_\theta + \sum_{i=1}^N e_{\gamma_i} \wedge e_{\gamma_{-i}} \right) \quad (\text{II.12})$$

satisfies CYBE and it will be called the extended Jordanian r -matrix of N -order. Let N be maximal order, i.e. there do not exist other elements $e_{\gamma_{\pm j}} \in \mathfrak{n}_+$, $j > N$, which satisfy the relations (II.11c) then the element (II.12) will be called the extended Jordanian r -matrix of maximal order [20]. It is evident that the extended Jordanian r -matrix of maximal order is defined by the elements $h_\theta \in \mathfrak{h}$, $e_\theta \in \mathfrak{n}_+$ and the Borel subalgebra \mathfrak{b}_+ . We shall here consider a special ("canonical") case when e_θ and $e_{\gamma_{\pm i}}$ ($i = 1, 2, \dots, N$) are weight elements with respect to the Cartan subalgebra \mathfrak{h}

$$[h, e_\theta] = (h, \theta) e_\theta, \quad [h, e_{\gamma_{\pm i}}] = (h, \gamma_{\pm i}) e_{\gamma_{\pm i}} \quad (\text{II.13})$$

for any $h \in \mathfrak{h}$ and for all $i = 1, 2, \dots, N$. Analyzing the structure of the positive root systems of the complex simple Lie algebras we see that if $e_{\gamma_{\pm j}} \in \mathfrak{n}_+$ the maximal order N of the extended Jordanian r -matrix is associated with the maximal root, i.e. the root θ is maximal.

Let us pass now to the Lie algebra $\mathfrak{g} = g_2$. The maximal root generator e_θ is $e_{2,3} = e_{2\alpha_1+3\alpha_2}$ and the extended Jordanian matrix of maximal order is provided by formula (II.12) with $N = 2$. It takes the form:

$$r_{2,3;2}(\xi) = \xi (h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0}). \quad (\text{II.14})$$

In order to obtain the generalizations of the r -matrix (II.14) one can use the theorem by Belavin and Drinfeld which states that the sum of two r -matrices r_1, r_2 is again a classical r -matrix [21] if r_2 has a carrier $L \in g_2$ ($r_2 \in L \otimes L$) which cocommutes with r_1 (i.e. it is a kernel of the bialgebra cobracket).

The maximal subalgebra in g_2 which is kernel of the Lie bialgebra cobracket determined by the r -matrix (II.14) has the following linear basis

$$L = (h_{0,1}, e_{0,1}, e_{0,-1}, e_{2,3}). \quad (\text{II.15})$$

i.e. $[r_{2,3;2}(\xi), l \otimes 1 + 1 \otimes l] = 0$ ($l \in L$). From the generators of the subalgebra L one can construct the following five classical r -matrices:

a) $h_{0,1} \wedge e_{0,1}$, b) $h_{0,1} \wedge e_{2,3}$, c) $e_{0,1} \wedge e_{2,3}$, d) $h_{0,1} \wedge e_{0,-1}$, e) $e_{0,-1} \wedge e_{2,3}$.

The r -matrices which we shall consider below are obtained as the linear combination of (II.14) and the r -matrices a) and b). One can show that the results of addition of the r -matrix (II.14) and the r -matrices c)–e) can be obtained from the previous two cases by suitable automorphisms of the algebra g_2 .

It follows that we can consider two r -matrices as basic ones, or more explicitly:

$$r_1 = \alpha h_{0,1} \wedge e_{0,1} + \xi (h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0}), \quad (\text{II.16a})$$

$$r_2 = \beta h_{0,1} \wedge e_{2,3} + \xi (h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0}), \quad (\text{II.16b})$$

where ξ, α, β are arbitrary.

One can raise the question whether the classical r -matrices (II.16a,b) can be extended to carrier space containing also the generators belonging to \mathfrak{b}_- . Unfortunately such an extension, which can not be eliminated by the inner automorphism of g_2 , is not possible from purely algebraic reason. One can show that there does not exist an even

dimensional subalgebra of g_2 , with dimension ten (two extra generators from \mathfrak{b}_-), which extends the full Borel subalgebra \mathfrak{b}_+ . In fact, the consideration of classical r -matrices with the carrier in both Borel subalgebras of g_2 which however are not simultaneously the classical r -matrices for $sl(3)$ subalgebra is an interesting problem to study, going beyond the scope of the present paper.

Below we shall consider the quantization of g_2 in the four steps, corresponding to the quantization of the following sequence of r -matrices:

i) Jordanian twist quantization

$$r_J = \xi h_{2,3} \wedge e_{2,3}. \quad (\text{II.17})$$

ii) Two extended Jordanian twist quantizations

$$r_{EJ} = \xi (h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2}), \quad (\text{II.18a})$$

$$r_{E'EJ} = \xi (h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0}). \quad (\text{II.18b})$$

The r -matrix r_{EJ} describes the extended Jordanian twist quantization of the $sl(3)$ subalgebra.

iii) Full twist quantization with additional twist factors describing deformed Jordanian twist (classical r -matrix (II.16a)) and the Abelian twist (classical r -matrix (II.16b)).

It should be observed that the parameters α, β and ξ occurring in the classical r -matrices (II.16a,b) can be rescaled by inner automorphisms of g_2 algebra as well as by the overall scaling of the r -matrices. In particular performing the two-parameter rescaling by Cartan generators (we use the notation $(ad^\otimes a)A \otimes B \equiv [a, A] \otimes B + A \otimes [a, B]$).

$$\begin{aligned} \exp[ad^\otimes(c_1 h_{1,0} + c_2 h_{0,1})]r_1 &= e^{\frac{1}{2}c_1}r_1, \\ \exp[ad^\otimes(c_1 h_{1,0} + c_2 h_{0,1})]r_2 &= \\ &= e^{(-\frac{1}{2}c_1 + \frac{1}{3}c_2)}\beta h_{0,1} \wedge e_{0,1} + e^{\frac{1}{2}c_1}r_{E'EJ} \end{aligned} \quad (\text{II.19})$$

we see that while the parameter α remains unchanged, the parameters β and ξ can be rescaled e.g. to unity. In order to modify the parameter α we can employ the overall scaling of the r -matrix. We see therefore, that similarly like in the case of Jordanian deformation of $sl(2)$ or κ -deformation of Poincaré algebra, the deformations with different values of the parameters α, β and ξ are mathematically equivalent (provided $\alpha \neq 0, \beta \neq 0, \xi \neq 0$) but distinguishable if applied to physical models.

III. TWIST QUANTIZATION METHOD AND THE GENERAL TWIST FUNCTIONS FOR g_2

A. Quantum deformations by twisting coproducts of Universal enveloping algebras

Consider the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} as a Hopf algebra with the comultiplication $\Delta^{(0)}$ generated by the primitive coproduct in \mathfrak{g} . The parametric invertible solution $\mathcal{F}(\xi) = \sum f_i^{(1)} \otimes f_i^{(2)} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ of the twist equations [8]

$$\mathcal{F}_{12}(\Delta^{(0)} \otimes 1)(\mathcal{F}) = \mathcal{F}_{23}(1 \otimes \Delta^{(0)})(\mathcal{F}), \quad (\text{III.1})$$

$$(\epsilon \otimes \text{id})(\mathcal{F}) = (\text{id} \otimes \epsilon)(\mathcal{F}) = 1 \otimes 1, \quad (\text{III.2})$$

defines the deformed (twisted) Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ with the unchanged multiplication, unit and counit (as in $U(\mathfrak{g})$), the twisted comultiplication and antipode defined by the relations

$$\Delta_{\mathcal{F}}(u) = \mathcal{F}\Delta^{(0)}(u)\mathcal{F}^{-1}, \quad u \in U(\mathfrak{g}), \quad (\text{III.3a})$$

$$S_{\mathcal{F}}(u) = v S^{(0)}(u) v^{-1}, \quad v = \sum f_i^{(1)} S^{(0)}(f_i^{(2)}). \quad (\text{III.3b})$$

The twisted algebra $U_{\mathcal{F}}(\mathfrak{g})$ is triangular, with the universal \mathcal{R} -matrix

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{F}^{-1}, \quad (\text{III.4})$$

which belongs to some extension of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. When \mathcal{F} is a smooth function of ξ and $\lim_{\xi \rightarrow 0} \mathcal{F} = 1 \otimes 1$ then in the neighborhood of the origin the \mathcal{R} -matrix can be presented as

$$\mathcal{R}_{\mathcal{F}} = 1 \otimes 1 + \xi r_{\mathcal{F}} + o(\xi), \quad (\text{III.5})$$

where $r_{\mathcal{F}}$ is the skewsymmetric classical r -matrix corresponding to the twist \mathcal{F} . Let us write explicitly the r -matrix as follows:

$$r_{\mathcal{F}} = a^{ij} I_i \wedge I_j. \quad (\text{III.6})$$

Then we obtain

$$\mathcal{F} = 1 \otimes 1 + \xi \tilde{a}^{ij} I_i \otimes I_j + \mathcal{O}(\xi), \quad (\text{III.7})$$

where $a^{ij} = \frac{1}{2}(\tilde{a}^{ij} - \tilde{a}^{ji})$.

By a nonlinear change of basis in $U(\mathfrak{g})$ one can modify the twisted coproducts and locate part of the deformation in the algebraic sector.

B. Twist deformations for $U(g_2)$ Hopf algebra

Our aim is to construct explicitly such a sequence of the twist deformations $U_{\mathcal{F}}(g_2)$ of the algebra $U(g_2)$ that will lead to the largest possible carrier subalgebra for the corresponding classical r -matrices. The final element of the corresponding twists will be the full chain of extended twists whose carrier coincides with the Borel subalgebra of g_2 . The peculiarity of the chain twist deformation is that the deformed algebra can be twisted step by step by the consecutive twisting factors with their specific properties. One of the important aims will be also the construction of proper nonlinear basis in $U(g_2)$. Indeed, on each step we shall construct the nonlinear basis in which the costructure of the Hopf algebra $U_{\mathcal{F}}(g_2)$ becomes more transparent.

In Sect. II we have presented the sequence of classical r -matrices for $U(g_2)$ (see (II.17), (II.18a,b) and (II.16a,b)). The quantization of these classical r -matrices is performed as follows.

a) Firstly we introduce the standard Jordanian twist quantizing the classical r -matrix (II.17), corresponding to the long root $2\alpha_1 + 3\alpha_2$ in g_2 . We have the following twisting element [22]

$$\mathcal{F}_J = e^{h_{2,3} \otimes \sigma_{2,3}} = e^{H \otimes \sigma}, \quad (\text{III.8})$$

where

$$H = h_{2,3} = 2h_{1,0} + 3h_{0,1}, \quad \sigma = \ln(1 + e_{2,3}). \quad (\text{III.9})$$

b) There are four types of the extension twisting factors that can be applied to $U_J(g_2)$ [12]:

$$\begin{aligned} \mathcal{F}_{E_+} &= e^{e_{1,1} \otimes e_{1,2} e^{-\frac{1}{2}\sigma}}, \\ \mathcal{F}_{E_-} &= e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}}, \\ \tilde{\mathcal{F}}_{E_+} &= e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}}, \\ \tilde{\mathcal{F}}_{E_-} &= e^{-e_{1,0} \otimes e_{1,3} e^{-\frac{1}{2}\sigma}}. \end{aligned} \quad (\text{III.10})$$

They can be composed to provide the following four types of the two-element extensions of (III.8)

$$\begin{aligned} \mathcal{F}_{E_{++}} &= e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{e_{1,1} \otimes e_{1,2} e^{-\frac{1}{2}\sigma}}, \\ \mathcal{F}_{E_{+-}} &= e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}}, \\ \mathcal{F}_{E_{-+}} &= e^{-e_{1,0} \otimes e_{1,3} e^{-\frac{1}{2}\sigma}} e^{e_{1,1} \otimes e_{1,2} e^{-\frac{1}{2}\sigma}}, \\ \mathcal{F}_{E_{--}} &= e^{-e_{1,0} \otimes e_{1,3} e^{-\frac{1}{2}\sigma}} e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}}. \end{aligned} \quad (\text{III.11})$$

One can note that exponential factors in the twists (III.11) commute with each other, and do not describe themselves the solutions of twist equations (III.1,III.2) with primitive coproduct $\Delta^{(0)}$. The four twists (III.11) lead to the

equivalent Hopf algebras however their coalgebra relations differ considerably. The most elegant result is obtained when the extension is chosen as follows

$$\mathcal{F}_E := \mathcal{F}_{E_{-+}} = \mathcal{F}_{E_-} \mathcal{F}_{E_+} = e^{-e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma} e^{e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma}, \quad (\text{III.12})$$

with the extended twist

$$\mathcal{F}_{EJ} := e^{-e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma} e^{e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma} e^{H \otimes \sigma}. \quad (\text{III.13})$$

It should be added that the products of twists $\mathcal{F}_{E_{\pm}} \mathcal{F}_J$ describe the twist quantization of $sl(3)$ subalgebra.

c) The additional Abelian twist factor ($h \equiv 3h_{0,1}$)

$$\mathcal{F}_A = e^{h \otimes \sigma}, \quad (\text{III.14})$$

that produces a kind of a "rotation" in the root space of g_2 , can enlarge the extended twist (III.13):

$$\mathcal{F}_{AEJ} := e^{h \otimes \sigma} e^{-e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma} e^{e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma} e^{H \otimes \sigma}. \quad (\text{III.15})$$

In such a way we obtain the quantization of the classical r -matrix (II.16b).

d) We can construct the chain of twists (see e.g. [9, 10]) for g_2 by additionally deforming the twisted $U_{EJ}(g_2)$ by the second link of the chain, which is the Jordanian factor:

$$\mathcal{F}_{J'} = e^{h \otimes \omega} \quad (\text{III.16})$$

with

$$\omega = \ln \left(1 + e_{0,1} + \frac{1}{2} (e_{1,2})^2 \right). \quad (\text{III.17})$$

This gives the quantization with the largest carrier

$$\mathcal{F}_{J'EJ} := e^{h \otimes \omega} e^{-e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma} e^{e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma} e^{H \otimes \sigma}. \quad (\text{III.18})$$

The twist function (III.18) describes the quantization of the classical r -matrix (II.16a). The twist (III.15) can also form the chain with $\mathcal{F}_{J''} = e^{h \otimes \omega''}$ (see Section V.C). But the Abelian twist factors \mathcal{F}_A and this new Jordanian factor are related by the formula $\mathcal{F}_{J''} \mathcal{F}_A = \mathcal{F}_{J'}$. This means that for any "rotated" extended twist \mathcal{F}_{AEJ} we get the unique chain (III.18).

In the Sections IV and V we shall present the deformed Hopf algebras $U_{\mathcal{F}}(g_2)$ with more details, discuss their properties and introduce the suitable bases of $U(g_2)$. In such a way we obtain the twist quantizations for g_2 with the largest carrier, which can be described by the chain of twists.

The following paragraph (III.C) will be devoted to the description of the mathematical framework, which permits to choose the basis in $U(g_2)$ with simplified coproduct formulae. Subsequently, in the paragraph III D we shall derive some new mathematical formulae simplifying the calculation of coproducts in Sect. IV and V.

C. Dual bases and simplification of coalgebra structure

Let $(\mathfrak{g}, \mathfrak{g}^{\#})$ be a coboundary Lie bialgebra. The dual Lie algebra $\mathfrak{g}^{\#}$ is determined by the r -matrix and can be written explicitly in the form of Lie coalgebra with the cocommutators $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$, ($a \in \mathfrak{g}$). Let $\mathfrak{G}^{\#}$ denote the dual group for $(\mathfrak{g}, \mathfrak{g}^{\#})$ – the universal covering Lie group with the Lie algebra $\mathfrak{g}^{\#}$. According to the quantum duality principle [13] the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ naturally treated as quantum algebra with respect to $U(\mathfrak{g})$ can be also considered as a quantum group with respect to $\text{Fun}(\mathfrak{G}^{\#})$: $U_{\mathcal{F}}(\mathfrak{g}) \approx \text{Fun}_{\xi}(\mathfrak{G}^{\#})$. The coproducts in $U_{\mathcal{F}}(\mathfrak{g})$ describe the deformed group multiplication law of the dual group $\mathfrak{G}^{\#}$. These multiplications are deformed due to the fact that generators in $\text{Fun}_{\xi}(\mathfrak{G}^{\#})$ are subject to the relations of $U(\mathfrak{g})$. The undeformed coproducts for $\text{Fun}(\mathfrak{G}^{\#})$ can be obtained by constructing the second classical limit $\text{Fun}_{\xi}(\mathfrak{G}^{\#}) \longrightarrow \text{Fun}(\mathfrak{G}^{\#})$ for the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ [26]. Among other important consequences the quantum duality prescribes the existence of two preferred bases for the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ [25]: the natural set of generators for \mathfrak{g} (usually they form the Cartan-Weil basis in \mathfrak{g}) and the basis natural for $\mathfrak{G}^{\#}$. The latter may be the exponential basis, generated by the Cartan-Weil basis in $\mathfrak{g}^{\#}$. Evidently, the costructure of $U_{\mathcal{F}}(\mathfrak{g})$ becomes transparent only in terms the group $\mathfrak{G}^{\#}$ that is in the dual group basis

or (keeping in mind the exponential map) in $\mathfrak{g}^\#$ -basis. For algebras of $\text{rank}(\mathfrak{g}) = 1$ one can use the \mathfrak{g} -coordinates for the multiplications as well as for the comultiplications. Starting with $\text{rank}(\mathfrak{g}) = 2$ (see for example $U_{\mathcal{F}}(sl(3))$ in [12]) some of the coproducts written in classical \mathfrak{g} -basis (natural \mathfrak{g} -coordinates) are complicated and it is difficult to study their properties. To pass from \mathfrak{g} - to $\mathfrak{g}^\#$ -basis it is necessary to compare the generators in $U_{\mathcal{F}}(\mathfrak{g})$ induced by the undeformed $U(\mathfrak{g})$ with those corresponding to $\text{Fun}(\mathfrak{G}^\#)$ or to its canonical dual $(\text{Fun}(\mathfrak{G}^\#))^* \approx U(\mathfrak{g}^\#)$. The general description of the corresponding algorithm and in particular the transformation of the Lie-algebra bases (i.e. the change of coordinates $\mathfrak{g} \longleftrightarrow \mathfrak{g}^\#$) was presented in [13]. In Section IV for the obtained quantum algebras $U_{\mathcal{F}}(\mathfrak{g})$ we shall use both \mathfrak{g} - and $\mathfrak{g}^\#$ -bases and demonstrate their role in the description of algebraic and coalgebraic properties of twist deformations.

D. New algebraic formulae for the similarity transformation of tensor products

Calculations of deformed coproducts rely on successive application of the following version of the Baker-Campbell-Hausdorff (BCH) formula (the adjoint action in terms of exponential map):

$$e^X A e^{-X} = \exp(\text{ad}_X) A \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_X^k A = \sum_{k=0}^{\infty} \frac{1}{k!} [X, [\dots [X, A] \dots]]. \quad (\text{III.19})$$

More exactly, since one works in the tensor product of two algebras $U \otimes U$, the BCH formula we need is as follows,

$$\begin{aligned} e^{X \otimes Y} (A \otimes B) e^{-X \otimes Y} &= \exp(\text{ad}_{X \otimes Y}) (A \otimes B) \\ &= \exp(\text{ad}_X \otimes Y) (A \otimes 1) \exp(X \otimes \text{ad}_Y) (1 \otimes B) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_X^k A \otimes Y^k \sum_{m=0}^{\infty} \frac{1}{m!} X^m \otimes \text{ad}_Y^m B. \end{aligned} \quad (\text{III.20})$$

The above expression is a product of two infinite series. Fortunately, in most of applications important for our study here both series truncate and become finite due to the fact that one acts in the enveloping algebra $U = U_g$ of some finite dimensional simple Lie algebra g and adjoint actions of non-Cartan elements are nilpotent. The other case which can be handled well occurs when one of the elements, say X , belongs to Cartan subalgebra. Then $\text{ad}_X A \equiv [X, A] = aA$ and $\text{ad}_X^k A = a^k A$ with $a \in \mathbb{C}$. In this case the first factor shrinks into the following simple expression

$$\sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_X^k A \otimes Y^k = A \otimes e^{aY}. \quad (\text{III.21})$$

We see that the complexity of calculations heavily depends on the degree of nilpotency. We should also notice that in applications one of the factors, say $Y = f(E)$ might have a functional form in terms of some generator E , e.g. corresponding to maximal root vector. We are particularly interested when $f(E) = \ln(1 + E)$ or $f(E) = e^E$.

When degree of nilpotency is lower than two, i.e. $\text{ad}_E^2 B \equiv [E, [E, B]] = 0$ one can use the following obvious expression $[f(E), B] = [E, B]f'(E)$ (where f' denotes derivative of f). This leads to

$$\sum_{m=0}^{\infty} \frac{1}{m!} X^m \otimes \text{ad}_{f(E)}^m B = 1 \otimes B + X \otimes [E, B]f'(E). \quad (\text{III.22})$$

Dealing with these simple technique permits to calculate almost all deformed coproducts except the one described by the last twist factor \mathcal{F}_C (see (V.22)). In order to complete calculations one needs more general and more sophisticated methods. For this purpose we have found two combinatorial expressions which turned out to be very useful.

If X and Y are two commuting elements and in addition $[X, B] = 0$ then

$$Y^X B Y^{-X} \equiv e^{X \ln Y} B e^{-X \ln Y} = \sum_{k=0}^{\infty} X^k \frac{\text{ad}_Y^k B}{k!} Y^{-k} \quad (\text{III.23})$$

where the sequence

$$x^{\underline{k}} \equiv \frac{\Gamma(x+1)}{\Gamma(x-k+1)} = x(x-1)\dots(x-k+1) = \sum_{m=1}^k s(k, m) x^m \quad (\text{III.24})$$

stands for the so-called lower (or falling) factorial polynomials and $s(n, k)$ are the Stirling numbers of the first kind.

Notice that if $[Y, B] = 0$ and $[Y, X] = 0$ one gets

$$Y^X B Y^{-X} \equiv e^{X \ln Y} B e^{-X \ln Y} = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln Y)^k \text{ad}_X^k B \quad (\text{III.25})$$

Another useful combinatorial formula is

$$e^{Y e^E} B e^{-Y e^E} = \sum_{k=0}^{\infty} \frac{\text{ad}_E^k B}{k!} q_k(Y e^E) = \sum_{m=0}^{\infty} \left(\sum_{k=m}^{\infty} \frac{\text{ad}_E^k B}{k!} S(k, m) \right) Y^m e^{mE} \quad (\text{III.26})$$

whenever E and B commute with Y . Here the sequence

$$q_k(x) = \sum_{m=0}^k S(k, m) x^m \quad (\text{III.27})$$

is given by the so-called Bell polynomials and $S(n, k)$ are Stirling numbers of the second kind. It should be also remarked that both families of polynomials, $x^{\underline{k}}$ and $q_k(x)$, belong to the wide class of so-called convolution polynomials known also as polynomial sequences of binomial type since, e.g.

$$q_k(x+y) = \sum_{m=0}^k \binom{k}{m} q_m(x) q_{k-m}(y) \quad (\text{III.28})$$

Such polynomials are extensively discussed in combinatorial analysis and umbral calculus (see [23],[24]). Here we have found operator analogs of some fundamental formulae involving polynomial sequences of binomial type. All above formulae can be checked by direct calculation in any order of the power expansion, the non-perturbative proof will be presented elsewhere.

Both formulae (III.23) and (III.26) can be adjusted to the particular situation for non-standard (Jordanian) and extended twists. For example, adapting to the case of Jordanian twist $\mathcal{F}_J = e^{H \otimes \omega}$ with H being a Cartan element: $[H, A] = aA$, $\omega = \ln(1+E)$, one gets

$$\mathcal{F}_J A \otimes B \mathcal{F}_J^{-1} = \sum_{k=0}^{\infty} H^k \otimes \frac{\text{ad}_E^k B}{k!} e^{-k\omega} (A \otimes e^{a\omega}) = \sum_{k=0}^{\infty} H^k A \otimes \frac{\text{ad}_E^k B}{k!} e^{(a-k)\omega}. \quad (\text{III.29})$$

Similarly, the counterpart of (III.29) for extended twist (double exponential) $\mathcal{F}_E = e^{X \otimes Y e^{f(E)}}$ takes the form

$$\begin{aligned} \mathcal{F}_E(A \otimes B) \mathcal{F}_E^{-1} &= \sum_{m=0}^{\infty} \frac{1}{m!} X^m \otimes \text{ad}_{f(E)}^m B q_m(Y e^{f(E)}) \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_X^k A \otimes Y^k e^{k f(E)} \\ &= \sum_{k, m \geq 0} \frac{1}{m! k!} X^m \text{ad}_X^k A \otimes \text{ad}_{f(E)}^m B \sum_{i=0}^m S(m, i) Y^{k+i} e^{(k+i)f(E)} \end{aligned} \quad (\text{III.30})$$

provided that $[Y, B] = [Y, E] = 0$.

In Sections IV and V these last formulae shall be particularly useful.

IV. TWIST QUANTIZATIONS OF g_2 GENERATED BY THE $sl(3)$ TWISTS.

We start by considering the two twist factors \mathcal{F} corresponding to the classical r -matrix (II.18a) with the carrier subalgebras inside the Borel subalgebra of $sl(3) \subset g_2$. This choice will permit to construct first two steps of the twist deformation corresponding to the r -matrices (II.16a,b). Performing such twisting one can compare deformed Hopf algebras $U_{\mathcal{F}}(g_2)$ with the known deformation $U_{\mathcal{F}}(sl(3))$ [8, 12].

A. The First Jordanian Twist

The Jordanian twist \mathcal{F}_J can be based on any 2-dimensional Borel subalgebra $\mathfrak{b}_+(sl(2)) \subset g_2$. In our case this subalgebra is generated by $\{H \equiv h_{2,3}, e_{2,3}\}$. The Jordanian r -matrix is given by (II.17), and the corresponding twisting element is $\mathcal{F}_J = e^{H \otimes \sigma}$ (see (III.8))

The costructure of the deformed algebra $U_J(g_2)$ is obtained by applying the similarity map (III.3a) to the primitive coproducts in $U(g_2)$. In the carrier subalgebra $\mathfrak{b}_+(sl(2))$ we obtain

$$\Delta_J(e_{2,3}) = e_{2,3} \otimes e^\sigma + 1 \otimes e_{2,3}, \quad \text{or} \quad \Delta_J(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma, \quad (\text{IV.1})$$

$$\Delta_J(H) = H \otimes e^{-\sigma} + 1 \otimes H. \quad (\text{IV.2})$$

The generators of the long root sequences $\pm(\alpha_1 + k\alpha_2)$ ($k = 0, 1, 2, 3$) have the coproducts

$$\Delta_J(e_{1,k}) = e_{1,k} \otimes e^{\frac{1}{2}\sigma} + 1 \otimes e_{1,k}, \quad (\text{IV.3})$$

$$\Delta_J(e_{-1,-k}) = e_{-1,-k} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,-k} + (-1)^k H \otimes e_{1,(3-k)} e^{-\sigma}, \quad (\text{IV.4})$$

describing 1- and 2- dimensional subrepresentations of the Borel subalgebra $\mathfrak{b}_+(sl(2))$. The generators with the roots "orthogonal" to $2\alpha_1 + 3\alpha_2$ remain primitive:

$$\Delta_J(h) = h \otimes 1 + 1 \otimes h, \quad (\text{IV.5})$$

$$\Delta_J(e_{0,\pm 1}) = e_{0,\pm 1} \otimes 1 + 1 \otimes e_{0,\pm 1}. \quad (\text{IV.6})$$

and describe 2-dimensional subrepresentations of $\mathfrak{b}_+(sl(2))$. One can check that in $\mathfrak{g}^\# \setminus \mathfrak{b}_+(sl(2))$ the generator H induces a shift $\text{ad}_H : e_{1,(3-k)} \longrightarrow e_{-1,-k}$ and this is exactly what indicates the last terms of the above coproducts $\Delta_J(e_{-1,-k})$.

The coproduct for the lowest root generator $e_{-2,-3}$,

$$\begin{aligned} \Delta_J(e_{-2,-3}) &= e_{-2,-3} \otimes e^{-\sigma} + 1 \otimes e_{-2,-3} \\ &+ (H - H^2) \otimes (e^{-\sigma} - e^{-2\sigma}) + 2H \otimes H e^{-\sigma}, \end{aligned} \quad (\text{IV.7})$$

also refers to a 1-dimensional subrepresentation. This can be seen when we pass to the $\mathfrak{g}^\#$ -basis. The following generator should be redefined here,

$$\widetilde{e_{-2,-3}} := e_{-2,-3} - H^2. \quad (\text{IV.8})$$

The coproduct $\Delta_J(\widetilde{e_{-2,-3}})$ is quasiprimitive and similar to $\Delta_J(H)$:

$$\Delta_J(\widetilde{e_{-2,-3}}) = \widetilde{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widetilde{e_{-2,-3}}. \quad (\text{IV.9})$$

For the case of twisted $U_J(sl(2))$ the nonlinear transformation (IV.8) was first indicated in [22].

B. The First Extended Jordanian Twist

The peculiarity of the chain twist deformation is that the deformed algebra can be twisted further by other twisting factors. The quantization goes step by step and on each level you get the deformed symmetry with its specific properties.

For the carrier $\mathfrak{g}^c \subset sl(3) \subset g_2$ the extension factor should be

$$\mathcal{F}_{E_+} = e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}}. \quad (\text{IV.10})$$

We remind that $sl(3)$ subalgebra is generated by the following 8 elements $\{H, h, e_{\pm 2, \pm 3}, e_{\pm 1, 0}, e_{\pm 1, \pm 3}\}$.

The twist (IV.10) is a solution of the twist equations (III.1), (III.2) for $U_J(g_2)$ and the adjoint operator $\exp\left(\text{ad}\left(e_{1,3} \otimes e_{1,0}e^{-\frac{1}{2}\sigma}\right)\right)$ applied to the coproducts in $U_J(g_2)$ will perform the deformation $\mathcal{F}_{E_+} : U_J(g_2) \longrightarrow U_{E_+J}(g_2)$. The same result can be obtained directly by applying the twist $\mathcal{F}_{E_+J} = (\mathcal{F}_{E_+}\mathcal{F}_J) : U(g_2) \longrightarrow U_{E_+J}(g_2)$.

In the costructure of $U_{E_+J}(g_2)$ we see the group multiplication of the solvable 4-dimensional Lie group (see [12]) with the Lie algebra equivalent to the carrier subalgebra $\mathfrak{g}_{E_+J}^c$ of the twist \mathcal{F}_{E_+J} :

$$\begin{aligned}\Delta_{E_+J}(e_{2,3}) &= e_{2,3} \otimes e^\sigma + 1 \otimes e_{2,3}, \\ \Delta_{E_+J}(H) &= H \otimes e^{-\sigma} + 1 \otimes H - e_{1,3} \otimes e_{1,0}e^{-\frac{3}{2}\sigma}, \\ \Delta_{E_+J}(e_{1,0}) &= e_{1,0} \otimes e^{\frac{1}{2}\sigma} + e^\sigma \otimes e_{1,0}, \\ \Delta_{E_+J}(e_{1,3}) &= e_{1,3} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{1,3}.\end{aligned}\tag{IV.11}$$

The other generators of the subalgebra $sl(3)$ form the 4-dimensional representation of \mathcal{L} :

$$\begin{aligned}\Delta_{E_+J}(h) &= h \otimes 1 + 1 \otimes h, \\ \Delta_{E_+J}(e_{-1,-3}) &= e_{-1,-3} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,-3} + h \otimes e_{1,0}e^{-\sigma}, \\ \Delta_{E_+J}(e_{-1,0}) &= +e_{-1,0} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,0} + H \otimes e_{1,3}e^{-\sigma} \\ &\quad - e_{1,3} \otimes e_{1,0}e_{1,3}e^{-\frac{3}{2}\sigma} + e_{1,3} \otimes (H - h)e^{-\frac{1}{2}\sigma} \\ &\quad - (e_{1,3})^2 \otimes e_{1,0}e^{-2\sigma} - He_{1,3} \otimes (e^\sigma - 1)e^{-\frac{3}{2}\sigma}, \\ \Delta_{E_+J}(\widetilde{e_{-2,-3}}) &= +\widetilde{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widetilde{e_{-2,-3}} - e_{-1,0} \otimes e_{1,0}e^{-\frac{3}{2}\sigma} \\ &\quad + He_{1,3} \otimes e_{1,0}e^{-\frac{3}{2}\sigma} + e_{1,3} \otimes e_{-1,-3}e^{-\frac{1}{2}\sigma}.\end{aligned}\tag{IV.12}$$

Finally on the remaining 6-dimensional space we also observe the adjoint action of the carrier subalgebra $\mathfrak{g}_{E_+J}^c$ ($k = 1, 2$)

$$\begin{aligned}\Delta_{E_+J}(e_{1,k}) &= e_{1,k} \otimes e^{\frac{1}{2}\sigma} + 1 \otimes e_{1,k}, \\ \Delta_{E_+J}(e_{0,\pm 1}) &= e_{0,\pm 1} \otimes 1 + 1 \otimes e_{0,\pm 1} + e_{1,\frac{5\pm 1}{2}} \otimes e_{1,\frac{1\pm 1}{2}}e^{-\frac{1}{2}\sigma}, \\ \Delta_{E_+J}(e_{-1,-k}) &= e_{-1,-k} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,-k} + (-1)^k H \otimes e_{1,(3-k)}e^{-\sigma} \\ &\quad - e_{(2-k),(5-2k)} \otimes e_{(k-1),(k-2)}e^{-\sigma} \\ &\quad - e_{1,3} \otimes e_{1,0}e_{1,(3-k)}e^{-\frac{3}{2}\sigma}.\end{aligned}\tag{IV.13}$$

First of all notice that $sl(3)$ generates the well known twisted algebra $U_{EJ}(sl(3))$ [8] which here is a Hopf subalgebra $U_{EJ}(sl(3)) \subset U_{E_+J}(g_2)$. Thus we obtain the intermediate $U_{\mathcal{F}}(sl(3))$ twist quantization inside $U_{\mathcal{F}}(f_2)$.

The nonprimitive terms in $\Delta_{E_+J}(e_{0,\pm 1})$ are in agreement with the structure of $\mathfrak{g}^\#$. They describe the action of the dual carrier group $G_c^\#$ in the 2-dimensional indecomposable representations. The third term in $\Delta_{E_+J}(H)$ (see (IV.11)) is just due to the Heisenberg subgroup in G_c^* .

The essential nonlinearities in the costructure are present in the last terms in $\Delta_{E_+J}(e_{-1,-k})$ as well as in $\Delta_{E_+J}(\widetilde{e_{-2,-3}})$ and $\Delta_{E_+J}(e_{-1,0})$. It should be noticed that the generator $\widetilde{e_{-2,-3}} = e_{-2,-3} - H^2$ is not modified. This is a common property of all the generators with roots opposite to the Jordanian carrier.

Let us return to the nontrivial terms in the costructure Δ_{E_+J} . Comparing $\Delta_{E_+J}(e_{-1,0})$, $\Delta_{E_+J}(e_{-1,-k})$ and $\Delta_{E_+J}(e_{0,\pm 1})$ with the canonical multiplication in $U(\mathfrak{g}^\#)$ we find the following $\mathfrak{g}^\#$ -basis:

$$\begin{aligned}\widetilde{e_{-1,0}} &= e_{-1,0} - He_{1,3}, \\ \widetilde{e_{0,-1}} &= e_{0,-1} - e_{1,0}e_{1,2}e^{-\sigma}, \\ \widetilde{e_{0,1}} &= e_{0,1} - e_{1,3}e_{1,1}, \\ \widetilde{e_{-1,-2}} &= e_{-1,-2} - He_{1,1}.\end{aligned}\tag{IV.14}$$

In these terms the action of G^c on the 6-dimensional space becomes transparent:

$$\begin{aligned}
\Delta_{E+J}(\widetilde{e_{0,1}}) &= \widetilde{e_{0,1}} \otimes 1 + 1 \otimes \widetilde{e_{0,1}} - e_{1,1} \otimes e_{1,3} e^{\frac{1}{2}\sigma}, \\
\Delta_{E+J}(\widetilde{e_{0,-1}}) &= \widetilde{e_{0,-1}} \otimes 1 + 1 \otimes \widetilde{e_{0,-1}} - e_{1,0} e^{-\sigma} \otimes e_{1,2} e^{-\frac{1}{2}\sigma}, \\
\Delta_{E+J}(e_{-1,-1}) &= e_{-1,-1} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,-1} - H \otimes e_{1,2} e^{-\sigma} \\
&\quad - e_{1,3} \otimes \widetilde{e_{0,-1}} e^{-\frac{1}{2}\sigma}, \\
\Delta_{E+J}(\widetilde{e_{-1,-2}}) &= \widetilde{e_{-1,-2}} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes \widetilde{e_{-1,-2}} - e_{1,1} \otimes H e^{\frac{1}{2}\sigma} \\
&\quad - \widetilde{e_{0,1}} \otimes e_{1,0} e^{-\sigma}, \\
\Delta_{E+J}(\widetilde{e_{-1,0}}) &= \widetilde{e_{-1,0}} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes \widetilde{e_{-1,0}} - e_{1,3} \otimes h e^{-\frac{1}{2}\sigma}, \\
\Delta_{E+J}(\widetilde{e_{-2,-3}}) &= \widetilde{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widetilde{e_{-2,-3}} \\
&\quad - \widetilde{e_{-1,0}} \otimes e_{1,0} e^{-\frac{3}{2}\sigma} + e_{1,3} \otimes e_{-1,-3} e^{-\frac{1}{2}\sigma}.
\end{aligned} \tag{IV.15}$$

V. TWIST DEFORMATIONS SPECIFIC TO g_2

A. The Full Extended Twist

The Jordanian twist \mathcal{F}_J (4.1) can be enlarged by the second extension factor \mathcal{F}_{E-} . Such an extension is the special property of g_2 root system. It does not exist for any other rank 2 simple Lie algebra. The following element is the solution of the twist equations for the Hopf algebra $U_{E+J}(g_2)$,

$$\mathcal{F}_{E-} = e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}}. \tag{V.1}$$

Together with the previously studied twist \mathcal{F}_{E+J} we obtain the full extended twist

$$\mathcal{F}_{EJ} = e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}} e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{H \otimes \sigma}. \tag{V.2}$$

The carrier subalgebra $\mathfrak{g}_{EJ}^c \approx \mathfrak{g}_{EJ}^{c\#}$ is 6-dimensional, it contains two Heisenberg subalgebras with common central element $e_{2,3}$ and the Cartan generator H . Applying the twist (V.2) to $U(g_2)$ or the second extension (V.1) to the Hopf algebra $U_{E+J}(g_2)$ (constructed in the previous Section) we obtain the new deformed costructure Δ_{E+J} . The coproducts $\Delta_{EJ} := \Delta_{E+J}$ for the generators of \mathfrak{g}_{EJ}^c describe the group multiplication in $G_{EJ}^{c\#}$ which is defined by the following relations:

$$\begin{aligned}
\Delta_{EJ}(e_{2,3}) &= e_{2,3} \otimes e^\sigma + 1 \otimes e_{2,3}, \\
\Delta_{EJ}(H) &= H \otimes e^{-\sigma} + 1 \otimes H - e_{1,3} \otimes e_{1,0} e^{-\frac{3}{2}\sigma} + e_{1,2} \otimes e_{1,1} e^{-\frac{3}{2}\sigma}.
\end{aligned} \tag{V.3}$$

$$\begin{aligned}
\Delta_{EJ}(e_{1,l}) &= e_{1,l} \otimes e^{\frac{1}{2}\sigma} + e^\sigma \otimes e_{1,l}, \quad l = 0, 1 \\
\Delta_{EJ}(e_{1,2+l}) &= e_{1,2+l} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{1,2+l},
\end{aligned} \tag{V.4}$$

These six coproducts are typical for the extended Jordanian twists [8]: each extension adds a summand in $\Delta_{E+J}(H)$, the constituent roots generators remain quasiprimitive and the generator σ deformed by the extension factors.

On the plane "orthogonal" to the initial root $\lambda_0 = 2\alpha + 3\beta$ we find only one primitive generator

$$\Delta_{EJ}(h) = h \otimes 1 + 1 \otimes h. \tag{V.5}$$

Other two coproducts are deformed:

$$\begin{aligned}
\Delta_{E+J}(e_{0,1}) &= e_{0,1} \otimes 1 + 1 \otimes e_{0,1} \\
&\quad - e_{1,2} \otimes e_{1,2} e^{-\frac{1}{2}\sigma} + \frac{1}{2} (e_{1,2})^2 \otimes (1 - e^{-\sigma}), \\
\Delta_{E+J}(e_{0,-1}) &= e_{0,-1} \otimes 1 + 1 \otimes e_{0,-1} \\
&\quad - \frac{4}{3} e_{1,1} \otimes e_{1,1} e^{-\frac{1}{2}\sigma} - \frac{2}{3} (e^\sigma - 1) \otimes (e_{1,1})^2 e^{-\sigma}.
\end{aligned} \tag{V.6}$$

The generators in the negative sector have quite complicated coproducts in $U_{EJ}(g_2)$. For example

$$\begin{aligned}
\Delta_{EJ}(\widetilde{e_{-2,-3}}) &= \widetilde{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widetilde{e_{-2,-3}} \\
&\quad - (e_{-1,0} - He_{1,3}) \otimes e_{1,0} e^{-\frac{3}{2}\sigma} + e_{1,3} \otimes e_{-1,-3} e^{-\frac{1}{2}\sigma} \\
&\quad - \left(e_{-1,-1} - \frac{1}{4}e_{1,2} + He_{1,2} \right) \otimes e_{1,1} e^{-\frac{3}{2}\sigma} \\
&\quad - \left(\frac{2}{3}e_{0,1} + \frac{1}{4}(e_{1,2})^2 \right) \otimes (e_{1,1})^2 e^{-2\sigma} \\
&\quad + e_{1,2} \otimes \left(e_{-1,-2} - \frac{1}{4}e_{1,1} e^{-\sigma} \right) e^{-\frac{1}{2}\sigma} \\
&\quad + \frac{1}{2}(e_{1,2})^2 \otimes \left(\frac{1}{2}(e_{1,1})^2 e^{-\sigma} + e_{0,-1} \right) e^{-\sigma} \\
&\quad + \frac{2}{9}e_{1,3} \otimes (e_{1,1})^3 e^{-\frac{5}{2}\sigma} - \frac{1}{6}(e_{1,2})^3 \otimes e_{1,0} e^{-\frac{3}{2}\sigma}.
\end{aligned} \tag{V.7}$$

Notice that in this expression we use the generators $\widetilde{e_{-2,-3}}$, i. e. we suppose that $\widetilde{e_{-2,-3}}$ will be appropriate for the deformed costructure not only in $U_{E+J}(g_2)$ but also in $U_{EJ}(g_2)$.

The coproducts for the elements $\{h, e_{0,\pm 1}, e_{-1,-k}, e_{-2,-3}; k = 0, \dots, 3\}$ describe the action of the carrier group $G_{EJ}^{\#}$ in the 8-dimensional subrepresentation.

To make this adjoint action transparent we perform the coordinate transformation $\mathfrak{g} \Rightarrow \mathfrak{g}^{\#}$ (see Section III.C). According to the algorithm presented in [12] the new basic elements are introduced:

$$\begin{aligned}
\widetilde{e_{0,1}} &= e_{0,1} + \frac{1}{2}(e_{1,2})^2, \\
\widetilde{e_{0,-1}} &= e_{0,-1} + \frac{2}{3}(e_{1,1})^2 e^{-\sigma}, \\
\widetilde{e_{-1,-3}} &= e_{-1,-3} + \frac{2}{9}(e_{1,1})^3 e^{-2\sigma}, \\
\widetilde{e_{-1,-2}} &= e_{-1,-2}, \\
\widetilde{e_{-1,-1}} &= e_{-1,-1} + He_{1,2}, \\
\widetilde{e_{-1,0}} &= e_{-1,0} - He_{1,3} + \frac{1}{6}(e_{1,2})^3, \\
\widetilde{e_{-2,-3}} &= \widetilde{e_{-2,-3}}.
\end{aligned} \tag{V.8}$$

We see that the element $e_{-1,-2}$ belongs to the $\mathfrak{g}^{\#}$ -basis for the group $G_{EJ}^{\#}$ while the generator $\widetilde{e_{-2,-3}}$ remains unchanged however we recall that on the previous step of quantization (in $U_{E+J}(g_2)$) it was nontrivially deformed. The coproducts of generators (V.8) are the following

$$\begin{aligned}
\Delta_{EJ}(\widetilde{e_{0,1}}) &= \widetilde{e_{0,1}} \otimes 1 + 1 \otimes \widetilde{e_{0,1}}, \\
\Delta_{EJ}(\widetilde{e_{0,-1}}) &= \widetilde{e_{0,-1}} \otimes 1 + 1 \otimes \widetilde{e_{0,-1}}, \\
\Delta_{EJ}(h) &= h \otimes 1 + 1 \otimes h \\
\Delta_{EJ}(\widetilde{e_{-1,-3}}) &= \left(\widetilde{e_{-1,-3}} \right) \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes \left(\widetilde{e_{-1,-3}} \right) \\
&\quad + 3h_{0,1} \otimes e_{1,0} e^{-\sigma} + \widetilde{e_{0,-1}} \otimes e_{1,1} e^{-\sigma}, \\
\Delta_{EJ}(e_{-1,-2}) &= e_{-1,-2} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes e_{-1,-2} - h_{0,1} \otimes e_{1,1} e^{-\sigma} \\
&\quad - \widetilde{e_{0,1}} \otimes e_{1,0} e^{-\sigma} + e_{1,2} \otimes \widetilde{e_{0,-1}} e^{-\frac{1}{2}\sigma}, \\
\Delta_{EJ}(\widetilde{e_{-1,-1}}) &= \widetilde{e_{-1,-1}} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes \widetilde{e_{-1,-1}} + e_{1,2} \otimes h_{0,1} e^{-\frac{1}{2}\sigma} \\
&\quad - e_{1,3} \otimes \left(\widetilde{e_{0,-1}} \right) e^{-\frac{1}{2}\sigma} + \frac{4}{3} \left(\widetilde{e_{0,1}} \right) \otimes e_{1,1} e^{-\sigma}, \\
\Delta_{EJ}(\widetilde{e_{-1,0}}) &= \widetilde{e_{-1,0}} \otimes e^{-\frac{1}{2}\sigma} + 1 \otimes \widetilde{e_{-1,0}} - e_{1,3} \otimes 3h_{0,1} e^{-\frac{1}{2}\sigma}
\end{aligned}$$

$$\begin{aligned}
\Delta_{EJ} \left(\widetilde{e_{-2,-3}} \right) &= \widetilde{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widetilde{e_{-2,-3}} \\
&\quad + e_{1,2} \otimes \widetilde{e_{0,1}} e^{-\frac{1}{2}\sigma}, \\
&\quad - \widetilde{e_{-1,0}} \otimes e_{1,0} e^{-\frac{3}{2}\sigma} + e_{1,3} \otimes \widetilde{e_{-1,-3}} e^{-\frac{1}{2}\sigma} \\
&\quad - \widetilde{e_{-1,-1}} \otimes e_{1,1} e^{-\frac{3}{2}\sigma} - \frac{2}{3} \widetilde{e_{0,1}} \otimes (e_{1,1})^2 e^{-2\sigma} \\
&\quad + e_{1,2} \otimes e_{-1,-2} e^{-\frac{1}{2}\sigma} + \frac{1}{2} (e_{1,2})^2 \otimes \widetilde{e_{0,-1}} e^{-\sigma}.
\end{aligned} \tag{V.9}$$

These coproducts correspond to the adjoint action of the algebra $G_{EJ}^{c\#}$ on the 8-dimensional space $G_2 \setminus \mathfrak{g}_{EJ}^c$. One can notice two subrepresentations on the subspaces spanned by $\{\widetilde{e_{0,\pm 1}}, h\}$ and $\{\widetilde{e_{-1,-k}}, \widetilde{e_{-2,-3}}\}$. On the subspace with the generators $\{\widetilde{e_{0,\pm 1}}, h\}$ we have the trivial factor-representation. This property means that on the plane orthogonal to the initial root we have the $sl(2)$ subalgebra with primitive generators $\{\widetilde{e_{0,\pm 1}}, h\}$. Such an effect was first described in [9].

In the deformation $U(g_2) \longrightarrow U_{EJ}(g_2)$ the costructure (V.3-V.4) on the carrier subalgebra is the extended Jordanian twist with two extension factors (as can be seen for example in $U_{EJ}(sl(4))$). The specific properties of g_2 become important in the negative sector (relations (V.9)) where the peculiarities of the root system induce additional terms. For example, the coproducts $\Delta_{EJ} \left(\widetilde{e_{-1,-1}} \right)$ and $\Delta_{EJ} \left(\widetilde{e_{-1,-2}} \right)$ contain the last two terms depending on the generators $\widetilde{e_{0,1}}$ and $\widetilde{e_{0,-1}}$.

Comparing the generators $\widetilde{e_{k,l}}$ in (V.8) with $\widetilde{e_{k,l}}$ (see (4.9) and (4.15)) one can see that most of them have different expressions in terms of the initial \mathfrak{g} -basis. The reason is that the twisting (V.2) not only deforms the group $G_{E+J}^\#$ but also changes its realization in terms of the initial g_2 generators.

B. The Full Chain of Twists for G_2 : Adding the Second Jordanian Twist

The existence of the subalgebra $U(sl(2)) \subset U_{EJ}(g_2)$ with primitive generators $\{\widetilde{e_{0,\pm 1}}, h\}$ shows that the Hopf algebra $U_{EJ}(g_2)$ can be additionally deformed by the second Jordanian twist. In other words the twist equations (III.1-III.2) with $\Delta^{(0)}$ replaced by Δ_{EJ} have the solution $\mathcal{F}_{J'} = e^{h \otimes \omega}$, where ω is given by (III.17), i.e. one can perform the transformation $\mathcal{F}_{J'} : U_{EJ}(g_2) \longrightarrow U_{J'EJ}(g_2) := U_C(g_2)$. The same result can be achieved by the chain of twists (III.18):

$$\mathcal{F}_C = \mathcal{F}_{J'EJ} = e^{h \otimes \omega} e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}} e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{H \otimes \sigma} \tag{V.10}$$

applied to the initial $U(g_2)$.

The carrier subalgebra \mathfrak{g}_C^c is the Borel subalgebra $\mathfrak{b}^+(g_2)$. Applying the twist (V.10) to $U(g_2)$ we obtain the deformed costructure $\Delta_{J'EJ} := \Delta_C$ corresponding to the maximal carrier subalgebra in g_2 . In the Hopf algebra $U_C(g_2)$ we have two σ -like generators,

$$\begin{aligned}
\Delta_C(\sigma) &= \sigma \otimes 1 + 1 \otimes \sigma, \\
\Delta_C(\omega) &= \omega \otimes 1 + 1 \otimes \omega,
\end{aligned} \tag{V.11}$$

and the ordinary form of the coproduct for the Cartan generator of $\mathfrak{b}sl(2)$ twisted by $\mathcal{F}_{J'} = e^{h \otimes \omega}$:

$$\Delta_C(h) = h \otimes e^{-\omega} + 1 \otimes h.$$

In the coproducts for the elements corresponding to the positive long sequence $\alpha_1 + k\alpha_2$ ($k = 0, 1, 2, 3$) one can trace the adjoint action of the algebra $\mathfrak{g}_C^\#$:

$$\begin{aligned}
\Delta_C(e_{1,3}) &= e_{1,3} \otimes e^{-\frac{1}{2}\sigma + \frac{3}{2}\omega} + 1 \otimes e_{1,3}, \\
\Delta_C(e_{1,2}) &= e_{1,2} \otimes e^{-\frac{1}{2}\sigma + \frac{1}{2}\omega} + 1 \otimes e_{1,2} - h \otimes e_{1,3} e^{-\omega},
\end{aligned}$$

$$\begin{aligned}
\Delta_C(e_{1,1}) &= e_{1,1} \otimes e^{\frac{1}{2}\sigma - \frac{1}{2}\omega} + e^\sigma \otimes e_{1,1} \\
&\quad - h e^\sigma \otimes \left(e_{1,2} e^{-\omega} + \frac{1}{2} e_{1,3} e^{-2\omega} \right) \\
&\quad + \frac{1}{2} h^2 e^\sigma \otimes e_{1,3} e^{-2\omega},
\end{aligned} \tag{V.12}$$

$$\begin{aligned}
\Delta_C(e_{1,0}) &= e_{1,0} \otimes e^{\frac{1}{2}\sigma + \frac{3}{2}\omega} + e^\sigma \otimes e_{1,0} \\
&\quad - h e^\sigma \otimes \left(e_{1,1} e^{-\omega} + \frac{1}{12} e_{1,3} e^{-3\omega} + \frac{1}{2} e_{1,2} e^{-2\omega} \right) \\
&\quad + \frac{1}{2} h^2 e^\sigma \otimes (e_{1,2} e^{-2\omega} + e_{1,3} e^{-3\omega}) \\
&\quad - \frac{1}{6} h^3 e^\sigma \otimes e_{1,3} e^{-3\omega}.
\end{aligned} \tag{V.13}$$

The terms corresponding to the powers of $\text{ad}(X_h)$ of the short root operator dual to h (such as $-\frac{1}{6}h^3 e^\sigma \otimes e_{1,3} e^{-3\omega}$ in the last row) are accompanied by the additional summands that will disappear when we pass (via the second classical limit) to the group costructure for $G_C^\#$. This is especially evident when $\Delta_C(H)$ is considered:

$$\begin{aligned}
\Delta_C(H) &= e^{\text{adh} \otimes \omega} \left(\begin{aligned} &H \otimes e^{-\sigma} + 1 \otimes H \\ &- e_{1,3} \otimes e_{1,0} e^{-\frac{3}{2}\sigma} + e_{1,2} \otimes e_{1,1} e^{-\frac{3}{2}\sigma} \end{aligned} \right) \\
&= H \otimes e^{-\sigma} + 1 \otimes H \\
&\quad + h \otimes \left(-\frac{1}{2} (e_{1,2})^2 e^{-\omega} - \frac{1}{2} e_{1,2} e_{1,3} e^{-2\omega} - \frac{1}{3} (e_{1,3})^2 e^{-3\omega} \right) \\
&\quad + \frac{1}{2} h^2 \otimes (e_{1,3} e_{1,2} e^{-2\omega} + (e_{1,3})^2 e^{-3\omega}) \\
&\quad + \frac{1}{6} h^3 \otimes \left(- (e_{1,3})^2 e^{-3\omega} \right) - e_{1,3} \otimes e_{1,0} e^{-\frac{3}{2}(\sigma - \omega)} \\
&\quad - h e_{1,3} \otimes \left(-e_{1,1} e^{-\frac{5}{2}\sigma + \frac{3}{2}\omega} - \frac{1}{12} e_{1,3} e^{-\frac{9}{2}\sigma + \frac{3}{2}\omega} - \frac{1}{2} e_{1,2} e^{-\frac{7}{2}\sigma + \frac{3}{2}\omega} \right) \\
&\quad - \frac{1}{2} h^2 e_{1,3} \otimes \left(e_{1,2} e^{-\frac{7}{2}\sigma + \frac{3}{2}\omega} + e_{1,3} e^{-\frac{9}{2}\sigma + \frac{3}{2}\omega} \right) \\
&\quad + \frac{1}{6} h^3 e_{1,3} \otimes e_{1,3} e^{-\frac{9}{2}\sigma + \frac{3}{2}\omega} \\
&\quad + e_{1,2} \otimes e_{1,1} e^{-\frac{3}{2}\sigma + \frac{1}{2}\omega} - h e_{1,2} \otimes e_{1,2} e^{-\frac{3}{2}\sigma - \frac{1}{2}\omega} \\
&\quad - \frac{1}{2} h e_{1,2} \otimes e_{1,3} e^{-\frac{3}{2}\sigma - \frac{3}{2}\omega} + \frac{1}{2} h^2 e_{1,2} \otimes e_{1,3} e^{-\frac{3}{2}\sigma - \frac{3}{2}\omega}.
\end{aligned} \tag{V.14}$$

One can show that in the second classical limit the large number of terms in (V.14) will disappear.

Let us turn now to the determination of $\mathfrak{g}_C^\#$ -basis in $U_C(g_2)$.

In the negative root sector \mathfrak{n}_- the element $\widetilde{\widetilde{e_{0,-1}}}$ must be evidently changed:

$$\widehat{e_{0,-1}} = \widetilde{\widetilde{e_{0,-1}}} - \frac{1}{3} h^2. \tag{V.15}$$

This is due to the fact that on the subspace generated by $\widetilde{\widetilde{e_{0,1}}}$, $\widetilde{\widetilde{e_{0,-1}}}$ and h the twist deformation performed by $\mathcal{F}_{J'}$ is an ordinary Jordanian deformation for the algebra $sl(2)$. The coproduct for $\widehat{e_{0,-1}}$ is quasiprimitive:

$$\Delta_C(\widehat{e_{0,-1}}) = \widehat{e_{0,-1}} \otimes e^{-\omega} + 1 \otimes \widehat{e_{0,-1}}. \tag{V.16}$$

The other coproducts for negative sector generators are quite complicated despite the fact that they are obtained in the improved (dual group $G_{EJ}^\#$) basis $\{\widetilde{\widetilde{x}}\}$. The factor $\mathcal{F}_{J'}$ changes the dual group

$$\mathcal{F}_{J'} : G_{EJ}^\# \longrightarrow G_C^\# \tag{V.17}$$

and the corresponding Lie algebra generators must be redefined. Using the technique demonstrated in Section III.D we get the set of new generators:

$$\widehat{e_{0,1}} = \widetilde{\widetilde{e_{0,1}}} = e_{0,1} + \frac{1}{2}(e_{1,2})^2 = e^\omega - 1; \quad (\text{V.18a})$$

$$\widehat{e_{0,-1}} = \widetilde{\widetilde{e_{0,-1}}} - \frac{1}{3}h^2 = e_{0,-1} + \frac{2}{3}(e_{1,1})^2 e^{-\sigma} - \frac{1}{3}h^2; \quad (\text{V.18b})$$

$$\begin{aligned} \widehat{e_{-1,0}} &= \widetilde{\widetilde{e_{-1,0}}} + e_{1,2} + h e_{1,3} = e_{-1,0} + (h - H) e_{1,3} \\ &\quad + \frac{1}{6}(e_{1,2})^3 + e_{1,2}; \end{aligned} \quad (\text{V.18c})$$

$$\begin{aligned} \widehat{e_{-1,-1}} &= \widetilde{\widetilde{e_{-1,-1}}} - \frac{4}{3}e_{1,1}e^{-\sigma+\omega} - \frac{1}{3}h e_{1,2} + \frac{1}{3}h^2 e_{1,3} \\ &= e_{-1,-1} + \left(H - \frac{1}{3}h\right) e_{1,2} - \frac{4}{3}e_{1,1}e^{-\sigma+\omega} + \frac{1}{3}h^2 e_{1,3}; \end{aligned} \quad (\text{V.18d})$$

$$\begin{aligned} \widehat{e_{-1,-2}} &= \widetilde{\widetilde{e_{-1,-2}}} + e_{1,0}e^{-\sigma+\omega} - \frac{1}{3}h^2 e_{1,2} \\ &= e_{-1,-2} + e_{1,0}e^{-\sigma+\omega} - \frac{1}{3}h^2 e_{1,2}; \end{aligned} \quad (\text{V.18e})$$

$$\widehat{e_{-1,-3}} = \widetilde{\widetilde{e_{-1,-3}}} = e_{-1,-3} + \frac{2}{9}(e_{1,1})^3 e^{-2\sigma}; \quad (\text{V.18f})$$

$$\begin{aligned} \widehat{e_{-2,-3}} &= \widetilde{\widetilde{e_{-2,-3}}} + \frac{2}{3}(e_{1,1})^2 e^{-2\sigma+\omega} - \frac{1}{6}h^2 (e_{1,2})^2 - \frac{2}{3}h \\ &= e_{-2,-3} - H^2 + \frac{2}{3}(e_{1,1})^2 e^{-2\sigma+\omega} - \frac{1}{6}h^2 (e_{1,2})^2 - \frac{2}{3}h; \end{aligned} \quad (\text{V.18g})$$

In the new basis we find fewer nonzero costructure constants. One of the new generators corresponds to the boundary of \mathfrak{n}_- and thus is quasiprimitive,

$$\Delta_C(\widehat{e_{-1,0}}) = \widehat{e_{-1,0}} \otimes e^{-\frac{1}{2}\sigma + \frac{3}{2}\omega} + 1 \otimes \widehat{e_{-1,0}} \quad (\text{V.19})$$

(notice that the other boundary element is $\widehat{e_{0,-1}}$ with the coproduct (V.16)).

The number of independent terms in the coproducts corresponds to the number of different decompositions of the vector $\beta \in \sum_{\mathfrak{n}_-}$ in terms of the $\mathfrak{g}_C^{\#}$ roots. This explains why in the sequence $e_{-\alpha_1 - k\alpha_2}$ ($k = 0, 1, 2, 3$) the number of terms rapidly increases with k :

$$\begin{aligned} \Delta_C(\widehat{e_{-1,-1}}) &= \widehat{e_{-1,-1}} \otimes e^{-\frac{1}{2}\sigma + \frac{1}{2}\omega} + 1 \otimes \widehat{e_{-1,-1}} \\ &\quad + h \otimes (\widehat{e_{-1,0}} + e_{1,3}) e^{-\omega} - e_{1,3} \otimes \widehat{e_{0,-1}} e^{-\frac{1}{2}\sigma + \frac{3}{2}\omega}; \end{aligned} \quad (\text{V.20a})$$

$$\begin{aligned} \Delta_C(\widehat{e_{-1,-2}}) &= \widehat{e_{-1,-2}} \otimes e^{-\frac{1}{2}\sigma - \frac{1}{2}\omega} + 1 \otimes \widehat{e_{-1,-2}} \\ &\quad + h \otimes \widehat{e_{-1,-1}} e^{-\omega} + \frac{1}{2}h^2 \otimes \widehat{e_{-1,0}} e^{-2\omega} + e_{1,2} \otimes \widehat{e_{0,-1}} e^{-\frac{1}{2}\sigma + \frac{1}{2}\omega} \\ &\quad - \frac{1}{2}h \otimes \widehat{e_{-1,0}} e^{-2\omega} - \frac{1}{3}h \otimes e_{1,2} e^{-\omega} - \frac{2}{3}(h - h^2) \otimes e_{1,3} e^{-2\omega}; \end{aligned} \quad (\text{V.20b})$$

$$\begin{aligned} \Delta_C(\widehat{e_{-1,-3}}) &= \widehat{e_{-1,-3}} \otimes e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} + 1 \otimes \widehat{e_{-1,-3}} \\ &\quad + h \otimes \left(\begin{aligned} &\widehat{e_{-1,-2}} e^{-\omega} + \frac{1}{3}\widehat{e_{-1,0}} e^{-3\omega} - \widehat{e_{0,-1}} e_{1,2} e^{-\omega} - \frac{4}{3}e_{1,1} e^{-\sigma-\omega} \\ &- \frac{1}{2}\widehat{e_{-1,-1}} e^{-2\omega} - \frac{1}{2}e_{1,2} e^{-2\omega} - \frac{1}{2}\widehat{e_{0,-1}} e_{1,3} e^{-2\omega} + \end{aligned} \right) \end{aligned}$$

$$\begin{aligned}
& +h^2 \otimes \left(\frac{1}{2}\widehat{e_{-1,-1}}e^{-2\omega} + \frac{1}{2}\widehat{e_{0,-1}}e_{1,3}e^{-2\omega} + \right. \\
& \left. + \frac{1}{6}\widehat{e_{1,2}}e^{-2\omega} - \frac{1}{3}\widehat{e_{1,3}}e^{-3\omega} - \frac{1}{2}\widehat{e_{-1,0}}e^{-3\omega} \right) \\
& +h^3 \otimes \left(\frac{1}{6}\widehat{e_{-1,0}}e^{-3\omega} + \frac{1}{3}\widehat{e_{1,3}}e^{-3\omega} \right) \\
& +h\widehat{e_{0,-1}} \otimes \left(-e_{1,2}e^{-2\omega} - \frac{1}{2}e_{1,3}e^{-3\omega} \right) \\
& +\widehat{e_{0,-1}} \otimes e_{1,1}e^{-\sigma-\omega} + \frac{1}{2}h^2\widehat{e_{0,-1}} \otimes e_{1,3}e^{-3\omega};
\end{aligned} \tag{V.20c}$$

In the twisted algebra $U_C(g_2)$ the group multiplication of $G_C^\#$ is nontrivially deformed. In particular the remaining coproduct $\Delta_C(\widehat{e_{-2,-3}})$ is the largest one: it has 23 terms that correspond to the adjoint action of the dual carrier group $G_C^{c\#}$ on the space of its 6-dimensional subrepresentation, and the other 28 terms do appear only due to the noncommutativity of the coordinates.

$$\begin{aligned}
\Delta_C(\widehat{e_{-2,-3}}) &= \widehat{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \widehat{e_{-2,-3}} \\
& -\widehat{e_{-1,0}} \otimes e_{1,0}e^{-\frac{3}{2}\sigma+\frac{3}{2}\omega} - \widehat{e_{-1,-1}} \otimes e_{1,1}e^{-\frac{3}{2}\sigma+\frac{1}{2}\omega} \\
& +e_{1,2} \otimes \widehat{e_{-1,-2}}e^{-\frac{1}{2}\sigma+\frac{1}{2}\omega} + e_{1,3} \otimes \widetilde{\widehat{e_{-1,-3}}}e^{-\frac{1}{2}\sigma+\frac{3}{2}\omega} \\
& +\frac{1}{2}(e_{1,2})^2 \otimes \widehat{e_{0,-1}}e^{-\sigma+\omega} \\
& +h \otimes \widehat{e_{-1,-1}}e_{1,2}e^{-\omega} - \frac{1}{2}h^2 \otimes \widehat{e_{-1,-1}}e_{1,3}e^{-2\omega} \\
& +he_{1,2} \otimes \widehat{e_{-1,-1}}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} + h\widehat{e_{-1,-1}} \otimes e_{1,2}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} \\
& +\frac{1}{2}h^2e_{1,3} \otimes \widehat{e_{-1,-1}}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} - \frac{1}{2}h^2\widehat{e_{-1,-1}} \otimes e_{1,3}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} \\
& +\frac{1}{2}h^2 \otimes \widehat{e_{-1,0}}e_{1,2}e^{-2\omega} + \frac{1}{2}h^2e_{1,3} \otimes \widehat{e_{0,-1}}e_{1,3}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} \\
& -\frac{1}{3}h^3 \otimes \widehat{e_{-1,0}}e_{1,3}e^{-3\omega} + h\widehat{e_{-1,0}} \otimes e_{1,1}e^{-\frac{3}{2}\sigma+\frac{1}{2}\omega} \\
& -\frac{1}{2}h^2\widehat{e_{-1,0}} \otimes e_{1,2}e^{-\frac{1}{2}\sigma+\frac{1}{2}\omega} + \frac{1}{2}h^2e_{1,2} \otimes \widehat{e_{-1,0}}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} \\
& +\frac{1}{6}h^3e_{1,3} \otimes \widehat{e_{-1,0}}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} + \frac{1}{6}h^3\widehat{e_{-1,0}} \otimes e_{1,3}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} \\
& +he_{1,3} \otimes \widehat{e_{-1,-2}}e^{-\frac{1}{2}\sigma+\frac{1}{2}\omega} - he_{1,3} \otimes \widehat{e_{0,-1}}e_{1,2}e^{-\frac{1}{2}\sigma+\frac{1}{2}\omega} \\
& -h \otimes \left(\frac{4}{3}e_{1,1}e_{1,3}e^{-\sigma-\omega} + \frac{3}{2}e_{1,2}e_{1,3}e^{-2\omega} + \frac{1}{3}(e_{1,2})^2e^{-\omega} \right. \\
& \quad \left. + \frac{4}{3}(e_{1,3})^2e^{-3\omega} + \frac{1}{2}\widehat{e_{-1,0}}e_{1,2}e^{-2\omega} \right. \\
& \quad \left. + \frac{2}{3}\widehat{e_{-1,0}}e_{1,3}e^{-3\omega} - \frac{1}{2}\widehat{e_{-1,-1}}e_{1,3}e^{-2\omega} \right) \\
& -he_{1,2} \otimes \left(\frac{2}{3}e_{1,2}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} + \frac{5}{12}e_{1,3}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} \right) \\
& -he_{1,3} \otimes \left(\frac{4}{3}e_{1,1}e^{-\frac{3}{2}\sigma+\frac{1}{2}\omega} + \frac{1}{2}e_{1,2}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} \right. \\
& \quad \left. - \frac{1}{3}\widehat{e_{-1,0}}e^{-\frac{1}{2}\sigma-\frac{3}{2}\omega} + \frac{1}{2}\widehat{e_{0,-1}}e_{1,3}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} \right. \\
& \quad \left. + \frac{1}{2}\widehat{e_{-1,-1}}e^{-\frac{1}{2}\sigma-\frac{1}{2}\omega} \right) \\
& +h^2 \otimes \left(\frac{1}{2}e_{1,2}e_{1,3}e^{-2\omega} + 2(e_{1,3})^2e^{-3\omega} + \widehat{e_{-1,0}}e_{1,3}e^{-3\omega} \right)
\end{aligned}$$

$$\begin{aligned}
& +h^2 e_{1,2} \otimes \left(\frac{1}{6} e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} + \frac{1}{2} \widehat{e_{-1,0}} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} \right) \\
& +h^2 e_{1,3} \otimes \left(\frac{1}{6} e_{1,2} e^{-\frac{1}{2}\sigma - \frac{1}{2}\omega} - \frac{1}{3} e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} - \frac{1}{2} \widehat{e_{-1,0}} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} \right) \\
& -\frac{1}{2} h^3 \otimes (e_{1,3})^2 e^{-3\omega} + \frac{1}{3} h^3 e_{1,3} \otimes e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} \\
& +h \widehat{e_{-1,0}} \otimes \left(\frac{1}{3} e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} + \frac{1}{2} e_{1,2} e^{-\frac{1}{2}\sigma - \frac{1}{2}\omega} \right) \\
& +\frac{1}{2} h \widehat{e_{-1,-1}} \otimes e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega} - \frac{1}{2} h^2 \widehat{e_{-1,0}} \otimes e_{1,3} e^{-\frac{1}{2}\sigma - \frac{3}{2}\omega}.
\end{aligned} \tag{V.21}$$

Comparing these results with those for other simple Lie algebras (deformed by the full chains of twists) we see that contrary e.g. to the $sl(n)$ case in the twisted algebra $U_C(G_2)$ the dual generator $\widehat{e_{-2,-3}}$ differs considerably from the "Jordanian" $\widetilde{e_{-2,-3}} = e_{-2,-3} - H^2$ and as well most of the coproducts in the negative sector are strongly deformed. In this situation the appropriate basis for the presentation of the costructure plays a very essential role. In the initial \mathfrak{g} -basis the decomposition of the coproduct $\Delta_C(e_{-2,-3})$ contains more than four hundred terms. Using the $\mathfrak{g}_{EJ}^\#$ -basis $\{\widetilde{x}\}$ we can reduce this number to 109 and finally in $\mathfrak{g}_C^\#$ -basis we get the expression (V.21) with 51 terms. The effective technique is needed to perform the corresponding calculations and this is where the modified BCH-formulas (III.29–III.30) presented in Subsection III.D are very useful. The expressions (V.20a–V.20c) and (V.21) were obtained with their help.

The full chain of twists (V.10) can be parametrized as follows:

$$\mathcal{F}_C = \mathcal{F}_J \mathcal{F}_E \mathcal{F}_J = e^{h \otimes \omega(\xi, \psi)} e^{-\xi e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma(\xi)} e^{\xi e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma(\xi)} e^{H \otimes \sigma(\xi)} \tag{V.22}$$

with

$$\begin{aligned}
\sigma(\xi) &= \ln(1 + \xi e_{2,3}), \\
\omega(\xi, \psi) &= \ln\left(1 + \psi e_{0,1} + \frac{1}{2} \xi \psi (e_{1,2})^2\right).
\end{aligned} \tag{V.23}$$

Thus the full chain leads to the 2-parameter set of Hopf algebras $U_C(G_2; \xi, \psi)$. In particular this parametrization provides the possibility to study the second classical limit for $U_C(G_2)$. The latter is obtained by scaling the generators $x \rightarrow \frac{1}{\varepsilon} x$ and by going to the limit $\varepsilon, \xi, \psi \rightarrow 0$, with finite values of $\frac{\xi}{\varepsilon} = \zeta, \frac{\psi}{\varepsilon} = \eta$. In such a limit we get the composition law of the dual group $G_C^\#$:

$$\begin{aligned}
\Delta_C^\#(H) &= H \otimes e^{-\sigma(\zeta)} + 1 \otimes H \\
&- \zeta e_{1,3} \otimes e_{1,0} e^{-\frac{3}{2}(\sigma(\zeta) - \omega(\zeta, \eta))} + \zeta e_{1,2} \otimes e_{1,1} e^{-\frac{3}{2}\sigma(\zeta) + \frac{1}{2}\omega(\zeta, \eta)} \\
&- \frac{1}{2} \zeta \eta h \otimes (e_{1,2})^2 e^{-\omega(\zeta, \eta)} - \zeta \eta h e_{1,2} \otimes e_{1,2} e^{-\frac{3}{2}\sigma(\zeta) - \frac{1}{2}\omega(\zeta, \eta)} \\
&+ \zeta \eta h e_{1,3} \otimes e_{1,1} e^{-\frac{5}{2}\sigma(\zeta) + \frac{3}{2}\omega(\zeta, \eta)} + \frac{1}{2} \zeta \eta^2 h^2 e_{1,2} \otimes e_{1,3} e^{-\frac{3}{2}\sigma(\zeta) - \frac{3}{2}\omega(\zeta, \eta)} \\
&+ \frac{1}{2} \zeta \eta^2 h^2 \otimes e_{1,3} e_{1,2} e^{-2\omega(\zeta, \eta)} - \frac{1}{2} \zeta \eta^2 h^2 e_{1,3} \otimes e_{1,2} e^{-\frac{7}{2}\sigma(\zeta) + \frac{3}{2}\omega(\zeta, \eta)} \\
&- \frac{1}{6} \zeta \eta^3 h^3 \otimes (e_{1,3})^2 e^{-3\omega(\zeta, \eta)} + \frac{1}{6} \zeta \eta^3 h^3 e_{1,3} \otimes e_{1,3} e^{-\frac{9}{2}\sigma(\zeta) + \frac{3}{2}\omega(\zeta, \eta)},
\end{aligned} \tag{V.24a}$$

$$\Delta_C^\#(h) = h \otimes e^{-\omega(\zeta, \eta)} + 1 \otimes h, \tag{V.24b}$$

$$\begin{aligned}
\Delta_C^\#(e_{1,0}) &= e_{1,0} \otimes e^{\frac{1}{2}\sigma(\zeta) - \frac{3}{2}\omega(\zeta, \eta)} + e^{\sigma(\zeta)} \otimes e_{1,0} \\
&- \eta h e^{\sigma(\zeta)} \otimes e_{1,1} e^{-\omega(\zeta, \eta)} + \frac{1}{2} \eta^2 h^2 e^{\sigma(\zeta)} \otimes e_{1,2} e^{\sigma(\zeta) - 2\omega(\zeta, \eta)} \\
&- \frac{1}{6} \eta^3 h^3 e^{\sigma(\zeta)} \otimes e_{1,3} e^{\sigma(\zeta) - 3\omega(\zeta, \eta)},
\end{aligned} \tag{V.24c}$$

$$\Delta_C^\#(e_{1,1}) = e_{1,1} \otimes e^{\frac{1}{2}\sigma(\zeta) - \frac{1}{2}\omega(\zeta, \eta)} + e^{\sigma(\zeta)} \otimes e_{1,1}$$

$$\begin{aligned}
& -\eta h e^{\sigma(\zeta)} \otimes e_{1,2} e^{\sigma(\zeta)-\omega(\zeta,\eta)} \\
& + \frac{1}{2} \eta^2 h^2 e^{\sigma(\zeta)} \otimes e_{1,3} e^{\sigma(\zeta)-2\omega(\zeta,\eta)}
\end{aligned} \tag{V.24d}$$

$$\Delta_C^\#(e_{1,2}) = e_{1,2} \otimes e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} + 1 \otimes e_{1,2} - \eta h \otimes e_{1,3} e^{-\omega(\zeta,\eta)}, \tag{V.24e}$$

$$\Delta_C^\#(e_{1,3}) = e_{1,3} \otimes e^{-\frac{1}{2}\sigma(\zeta)+\frac{3}{2}\omega(\zeta,\eta)} + 1 \otimes e_{1,3}, \tag{V.24f}$$

$$\Delta_C^\#(e_{2,3}) = e_{2,3} \otimes e^{\sigma(\zeta)} + 1 \otimes e_{2,3}, \tag{V.24g}$$

$$\Delta_C^\#(\omega(\zeta, \eta)) = \omega(\zeta, \eta) \otimes 1 + 1 \otimes \omega(\zeta, \eta), \tag{V.24h}$$

$$\Delta_C^\#(\widehat{e_{0,-1}}) = \widehat{e_{0,-1}} \otimes e^{-\omega(\zeta,\eta)} + 1 \otimes \widehat{e_{0,-1}}, \tag{V.24i}$$

$$\Delta_C^\#(\widehat{e_{-1,0}}) = \widehat{e_{-1,0}} \otimes e^{-\frac{1}{2}\sigma(\zeta)+\frac{3}{2}\omega(\zeta,\eta)} + 1 \otimes \widehat{e_{-1,0}}, \tag{V.24j}$$

$$\begin{aligned}
\Delta_C^\#(\widehat{e_{-1,-1}}) &= \widehat{e_{-1,-1}} \otimes e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} + 1 \otimes \widehat{e_{-1,-1}} \\
&+ \eta h \otimes \widehat{e_{-1,0}} e^{-\omega(\zeta,\eta)} - \zeta e_{1,3} \otimes \widehat{e_{0,-1}} e^{-\frac{1}{2}\sigma(\zeta)+\frac{3}{2}\omega(\zeta,\eta)},
\end{aligned} \tag{V.24k}$$

$$\begin{aligned}
\Delta_C^\#(\widehat{e_{-1,-2}}) &= \widehat{e_{-1,-2}} \otimes e^{-\frac{1}{2}\sigma(\zeta)-\frac{1}{2}\omega(\zeta,\eta)} + 1 \otimes \widehat{e_{-1,-2}} \\
&+ \eta h \otimes \widehat{e_{-1,-1}} e^{-\omega(\zeta,\eta)} + \frac{1}{2} \eta^2 h^2 \otimes \widehat{e_{-1,0}} e^{-2\omega(\zeta,\eta)} \\
&+ \zeta e_{1,2} \otimes \widehat{e_{0,-1}} e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)},
\end{aligned} \tag{V.24l}$$

$$\begin{aligned}
\Delta_C^\#(\widehat{e_{-1,-3}}) &= \widehat{e_{-1,-3}} \otimes e^{-\frac{1}{2}\sigma(\zeta)-\frac{3}{2}\omega(\zeta,\eta)} + 1 \otimes \widehat{e_{-1,-3}} \\
&+ h \otimes \left(\eta \widehat{e_{-1,-2}} e^{-\omega(\zeta,\eta)} - \zeta \eta \widehat{e_{0,-1}} e_{1,2} e^{-\omega(\zeta,\eta)} \right) \\
&+ h^2 \otimes \left(\frac{1}{2} \eta^2 \widehat{e_{-1,-1}} e^{-2\omega(\zeta,\eta)} + \frac{1}{2} \zeta \eta^2 \widehat{e_{0,-1}} e_{1,3} e^{-2\omega(\zeta,\eta)} \right) \\
&+ \frac{1}{6} \eta^3 h^3 \otimes \widehat{e_{-1,0}} e^{-3\omega(\zeta,\eta)} - \zeta \eta h \widehat{e_{0,-1}} \otimes e_{1,2} e^{-2\omega(\zeta,\eta)} \\
&+ \zeta \widehat{e_{0,-1}} \otimes e_{1,1} e^{-\sigma(\zeta)-\omega(\zeta,\eta)} + \frac{1}{2} \zeta \eta^2 h^2 \widehat{e_{0,-1}} \otimes e_{1,3} e^{-3\omega(\zeta,\eta)},
\end{aligned} \tag{V.24m}$$

$$\begin{aligned}
\Delta_C^\#(\widehat{e_{-2,-3}}) &= \widehat{e_{-2,-3}} \otimes e^{-\sigma(\zeta)} + 1 \otimes \widehat{e_{-2,-3}} \\
&- \zeta \widehat{e_{-1,0}} \otimes e_{1,0} e^{-\frac{3}{2}\sigma(\zeta)+\frac{3}{2}\omega(\zeta,\eta)} + \zeta e_{1,3} \otimes \widehat{e_{-1,-3}} e^{-\frac{1}{2}\sigma(\zeta)+\frac{3}{2}\omega(\zeta,\eta)} \\
&- \zeta \widehat{e_{-1,-1}} \otimes e_{1,1} e^{-\frac{3}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} + \zeta e_{1,2} \otimes \widehat{e_{-1,-2}} e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} \\
&+ h \otimes \zeta \eta \widehat{e_{-1,-1}} e_{1,2} e^{-\omega(\zeta,\eta)} - \frac{1}{2} h^2 \otimes \zeta \eta^2 \widehat{e_{-1,-1}} e_{1,3} e^{-2\omega(\zeta,\eta)} \\
&+ \zeta \eta h e_{1,2} \otimes \widehat{e_{-1,-1}} e^{-\frac{1}{2}\sigma(\zeta)-\frac{1}{2}\omega(\zeta,\eta)} + \zeta \eta h \widehat{e_{-1,-1}} \otimes e_{1,2} e^{-\frac{1}{2}\sigma(\zeta)-\frac{1}{2}\omega(\zeta,\eta)} \\
&+ \frac{1}{2} \zeta \eta^2 h^2 e_{1,3} \otimes \widehat{e_{-1,-1}} e^{-\frac{1}{2}\sigma(\zeta)-\frac{1}{2}\omega(\zeta,\eta)}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\zeta\eta^2h^2\widehat{e_{-1,-1}} \otimes e_{1,3}e^{-\frac{1}{2}\sigma(\zeta)-\frac{3}{2}\omega(\zeta,\eta)} \\
& +\frac{1}{2}h^2 \otimes \zeta\eta^2\widehat{e_{-1,0}}e_{1,2}e^{-2\omega(\zeta,\eta)} + \frac{1}{2}\zeta^2\eta^2h^2e_{1,3} \otimes \widehat{e_{0,-1}}e_{1,3}e^{-\frac{1}{2}\sigma(\zeta)-\frac{1}{2}\omega(\zeta,\eta)} \\
& -\frac{1}{3}h^3 \otimes \zeta\eta^3\widehat{e_{-1,0}}e_{1,3}e^{-3\omega(\zeta,\eta)} + h\widehat{e_{-1,0}} \otimes \zeta\eta e_{1,1}e^{-\frac{3}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} \\
& -\frac{1}{2}\zeta\eta^2h^2\widehat{e_{-1,0}} \otimes e_{1,2}e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} + \frac{1}{2}\zeta\eta^2h^2e_{1,2} \otimes \widehat{e_{-1,0}}e^{-\frac{1}{2}\sigma(\zeta)-\frac{3}{2}\omega(\zeta,\eta)} \\
& +\frac{1}{6}\zeta\eta^3h^3e_{1,3} \otimes \widehat{e_{-1,0}}e^{-\frac{1}{2}\sigma(\zeta)-\frac{3}{2}\omega(\zeta,\eta)} + \frac{1}{6}\zeta\eta^3h^3\widehat{e_{-1,0}} \otimes e_{1,3}e^{-\frac{1}{2}\sigma(\zeta)-\frac{3}{2}\omega(\zeta,\eta)} \\
& +\zeta\eta he_{1,3} \otimes \widehat{e_{-1,-2}}e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} \\
& -\zeta^2\eta he_{1,3} \otimes \widehat{e_{0,-1}}e_{1,2}e^{-\frac{1}{2}\sigma(\zeta)+\frac{1}{2}\omega(\zeta,\eta)} + \frac{1}{2}\zeta^2(e_{1,2})^2 \otimes \widehat{e_{0,-1}}e^{-\sigma(\zeta)+\omega(\zeta,\eta)}.
\end{aligned} \tag{V.24n}$$

We have finished the construction of the universal enveloping algebra $U_C(g_2)$ twisted by the full chain of extended twists. The sequence of factors (V.10) cannot be essentially enlarged due to the absence of the Lie-Frobenius subalgebras in g_2 that nontrivially contain $\mathfrak{b}_+(g_2)$ (see the discussion in Section II). It is certainly possible to perform the additional Abelian twist $\mathcal{F}_{A'} = e^{\sigma \otimes \omega}$, but in such a case the carrier $\mathfrak{g}_C^\#$ is not changed. It appears that adding the twist $\mathcal{F}_{A'}$ leads to the different realization of the same $\mathfrak{g}_C^\#$ in terms of the generators of $U_C(g_2)$.

When such change of the realization of the carrier happens in the intermediate steps of the quantization this can lead to interesting results. We shall study this possibility in the next Subsection.

C. General Form of Extended Twist for g_2 : Additional Abelian Twist

Let us construct the quantization for the r -matrix (II.16b). As it was indicated in Section III this can be performed by the twisting element

$$\mathcal{F}_{AE}(\rho) = e^{\rho h \otimes \sigma} e^{-e_{1,2} \otimes e_{1,1}} e^{-\frac{1}{2}\sigma} e^{e_{1,3} \otimes e_{1,0}} e^{-\frac{1}{2}\sigma} e^{H \otimes \sigma}. \tag{V.25}$$

Here the parameter ρ is written explicitly because of its specific role. Notice that for any fixed ρ the element \mathcal{F}_{AE} depends on the deformation parameter ξ (due to the ξ -dependence of σ (see (V.22)). Studying the parametrized set $\{\mathcal{F}_{AE}(\xi, \rho)\}$ we are dealing with the family of twists indexed by the parameter ρ . We shall demonstrate that contrary to the case of the deformation parameter ξ where we get equivalent deformed algebras $U_{\mathcal{F}_{AE}(\xi, \rho)} \sim U_{\mathcal{F}_{AE}(\xi', \rho)}$ there are nonzero values of ρ for which the twisted algebras are inequivalent $U_{\mathcal{F}_{AE}(\xi, \rho)} \not\sim U_{\mathcal{F}_{AE}(\xi, \rho')}$.

In order to construct the twisted Hopf algebra $U_{\mathcal{F}_{AE}(\rho)}(g_2)$ we use the coproducts Δ_{EJ} (5.18-5.21) of the full extended twisting (5.3) obtained in Subsection V.A and apply the transformation $e^{\rho ad h \otimes \sigma}$ corresponding to the Abelian twist factor $\mathcal{F}_A = e^{\rho h \otimes \sigma}$. Each nontrivial twisting factor induces the transformation of the dual group and the new dual coordinates are to be constructed. Omitting the intermediate steps we present the results in terms of the new dual $\mathfrak{g}_{AE}^\#$ -basis:

$$\begin{aligned}
\overline{H} &= H + \rho h, \\
\overline{e_{-2,-3}} &= \widetilde{e_{-2,-3}} + \rho^2 h^2, \\
\overline{e_{-1,0}} &= \widetilde{e_{-1,0}} - \rho h e_{1,3}, \\
\overline{e_{-1,-1}} &= \widetilde{e_{-1,-1}} + \rho h e_{1,2}
\end{aligned} \tag{V.26}$$

(the other $\mathfrak{g}_E^\#$ -coordinates \widetilde{x} remain unchanged).

According to the properties of the twist \mathcal{F}_A the costructure on its carrier is conserved:

$$\begin{aligned}
\Delta_{AE}(h) &= h \otimes 1 + 1 \otimes h, \\
\Delta_{AE}(e_{2,3}) &= e_{2,3} \otimes e^\sigma + 1 \otimes e_{2,3}.
\end{aligned} \tag{V.27}$$

The changes in $\Delta_{AE}(\overline{H})$ and in the long sequence $\Delta_{AE}(e_{1,l})$ ($l = 0, 1, 2, 3$) are correlated with the root structure of the extensions:

$$\begin{aligned}
\Delta_{AE}(\overline{H}) &= (\overline{H}) \otimes e^{-\sigma} + 1 \otimes (\overline{H}) \\
&\quad - e_{1,3} \otimes e_{1,0} e^{(\rho(\alpha+3\beta)(h)-\frac{3}{2})\sigma} + e_{1,2} \otimes e_{1,1} e^{(\rho(\alpha+2\beta)(h)-\frac{3}{2})\sigma} \\
&= (\overline{H}) \otimes e^{-\sigma} + 1 \otimes (\overline{H}) \\
&\quad - e_{1,3} \otimes e_{1,0} e^{\frac{3}{2}(\rho-1)\sigma} + e_{1,2} \otimes e_{1,1} e^{\frac{1}{2}(\rho-3)\sigma}.
\end{aligned} \tag{V.28}$$

$$\begin{aligned}
\Delta_{AE}(e_{1,k}) &= e_{1,0} \otimes e^{(\frac{1}{2}+\rho(\alpha+k\beta)(h))\sigma} + e^\sigma \otimes e_{1,0}, \quad k=0,1 \\
\Delta_{AE}(e_{1,m}) &= e_{1,0} \otimes e^{(-\frac{1}{2}+\rho(\alpha+m\beta)(h))\sigma} + 1 \otimes e_{1,0}, \quad m=2,3
\end{aligned}$$

On the plane orthogonal to the highest root we find the quasiprimitive costructure:

$$\begin{aligned}
\Delta_{AE}(\widetilde{\widetilde{e_{0,1}}}) &= \widetilde{\widetilde{e_{0,1}}} \otimes e^{\rho\sigma} + 1 \otimes \widetilde{\widetilde{e_{0,1}}}, \\
\Delta_{AE}(\widetilde{\widetilde{e_{0,-1}}}) &= \widetilde{\widetilde{e_{0,-1}}} \otimes e^{-\rho\sigma} + 1 \otimes \widetilde{\widetilde{e_{0,-1}}}.
\end{aligned} \tag{V.29}$$

In the coproducts for the other basic elements (belonging to \mathfrak{n}_-) do appear the additional terms proportional to ρ

$$\begin{aligned}
\Delta_{AE}(\widetilde{\widetilde{e_{-1,-3}}}) &= \widetilde{\widetilde{e_{-1,-3}}} \otimes e^{-\frac{1}{2}(1+3\rho)\sigma} \\
&\quad + 1 \otimes \widetilde{\widetilde{e_{-1,-3}}} + (1-\rho)h \otimes e_{1,0}e^{-\sigma} + \widetilde{\widetilde{e_{0,-1}}} \otimes e_{1,1}e^{-(1+\rho)\sigma},
\end{aligned} \tag{V.30a}$$

$$\begin{aligned}
\Delta_{AE}(e_{-1,-2}) &= e_{-1,-2} \otimes e^{-\frac{1}{2}(1+\rho)\sigma} + 1 \otimes e_{-1,-2} \\
&\quad + \left(\rho - \frac{1}{3}\right)h \otimes e_{1,1}e^{-\sigma} - \widetilde{\widetilde{e_{0,1}}} \otimes e_{1,0}e^{(\rho-1)\sigma} \\
&\quad + e_{1,2} \otimes \widetilde{\widetilde{e_{0,-1}}}e^{-\frac{1}{2}\sigma(1-\rho)},
\end{aligned} \tag{V.30b}$$

$$\begin{aligned}
\Delta_{AE}(\overline{e_{-1,-1}}) &= \overline{e_{-1,-1}} \otimes e^{\frac{1}{2}(\rho-1)\sigma} + 1 \otimes \overline{e_{-1,-1}} \\
&\quad + \left(\frac{1}{3} + \rho\right)e_{1,2} \otimes h e^{\frac{1}{2}(\rho-1)\sigma} \\
&\quad - e_{1,3} \otimes \widetilde{\widetilde{e_{0,-1}}}e^{\frac{1}{2}(3\rho-1)\sigma} + \frac{4}{3}\widetilde{\widetilde{e_{0,1}}} \otimes e_{1,1}e^{(\rho-1)\sigma},
\end{aligned} \tag{V.30c}$$

$$\begin{aligned}
\Delta_{AE}(\overline{e_{-1,0}}) &= \overline{e_{-1,0}} \otimes e^{\frac{1}{2}(3\rho-1)\sigma} + 1 \otimes \overline{e_{-1,0}} + e_{1,2} \otimes \widetilde{\widetilde{e_{0,1}}}e^{-\frac{1}{2}\sigma(1-\rho)} \\
&\quad - (\rho+1)e_{1,3} \otimes h e^{\frac{1}{2}(3\rho-1)\sigma},
\end{aligned} \tag{V.30d}$$

$$\begin{aligned}
\Delta_{AE}(\overline{e_{-2,-3}}) &= \overline{e_{-2,-3}} \otimes e^{-\sigma} + 1 \otimes \overline{e_{-2,-3}} + 2\rho h \otimes \overline{H} \\
&\quad - \overline{e_{-1,0}} \otimes e_{1,0}e^{\frac{3}{2}(\rho-1)\sigma} - \overline{e_{-1,-1}} \otimes e_{1,1}e^{\frac{1}{2}(\rho-3)\sigma} \\
&\quad + e_{1,2} \otimes e_{-1,-2}e^{\frac{1}{2}(\rho-1)\sigma} \\
&\quad + e_{1,3} \otimes \widetilde{\widetilde{e_{-1,-3}}}e^{\frac{1}{2}(3\rho-1)\sigma} - \frac{2}{3}\widetilde{\widetilde{e_{0,1}}} \otimes (e_{1,1})^2 e^{(\rho-2)\sigma} \\
&\quad + \frac{1}{2}(e_{1,2})^2 \otimes \widetilde{\widetilde{e_{0,-1}}}e^{(\rho-1)\sigma} + 2\rho h e_{1,2} \otimes e_{1,1}e^{\frac{1}{2}(\rho-3)\sigma}.
\end{aligned} \tag{V.30e}$$

In the last expression the additional terms signify that the root vector \overline{H}^* is no longer orthogonal to h^* and the adjoint operator $\text{ad}(\overline{H}^*)$ transforms h^* into $\overline{e_{-2,-3}}$.

In comparison with $\Delta_E(\widetilde{\widetilde{x}})$ the changes of the form of the coproducts Δ_{AE} are small but essential. We immediately see five singular points $\rho = 0, \pm 1, \pm \frac{1}{3}$. In each of them one of the coproducts loses some terms and becomes closer to a quasiprimitive. For example in the cases $\Delta_{AE_+}(\widetilde{\widetilde{e_{-1,-3}}})$ and $\Delta_{AE_+}(\overline{e_{-1,0}})$ if we use only the $sl(3)$ -extension

$\mathcal{F}_+ = e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}}$ then for $\rho = \pm 1$ these coproducts become quasiprimitive and we obtain the possibility to perform additional twistings with the carrier algebra nontrivially intersecting with \mathfrak{n}_- . In the costructure Δ_{AE} the corresponding enlargement of the carrier cannot be achieved, nevertheless the singular points are also important. In particular we see that the standard case Δ_{EJ} corresponds to the singular point $\rho = 0$ while in the general situation the coproduct $\Delta_{AE(\rho)}(\overline{e_{-2,-3}})$ has additional components and we arrive at different dual algebra $\mathfrak{g}_{AE}^\# \neq \mathfrak{g}_E^\#$. Notice that in all the singular points the corresponding r -matrices differ only by the value of the numerical parameter ρ but the results of the quantizations are different and refer to Lie-Poisson structures.

We have already seen that the coproducts $\Delta_{AE}(\widetilde{e_{0,\pm 1}})$ are now quasiprimitive. Consequently there still exists the possibility to perform further twisting with the Jordanian factor similar to $\mathcal{F}_{J'}$

$$\mathcal{F}_{J''} = e^{h \otimes \omega'} \quad (\text{V.31})$$

with $\omega' = \ln(\widetilde{e_{0,1}} e^{-\rho\sigma} + e^{-\rho\sigma})$. The twisting element

$$\mathcal{F}_{J''AE} = e^{h \otimes \omega'} e^{\rho h \otimes \sigma} e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}} e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{H \otimes \sigma} \quad (\text{V.32})$$

is the solution of the twist equation. Here, contrary to the situation with the full chain of extended twists, we do not obtain a parametrized family of chain deformations. It can be easily checked that (see also (V.10))

$$\mathcal{F}_{J''AE} = \mathcal{F}_C = e^{h \otimes \omega} e^{-e_{1,2} \otimes e_{1,1} e^{-\frac{1}{2}\sigma}} e^{e_{1,3} \otimes e_{1,0} e^{-\frac{1}{2}\sigma}} e^{H \otimes \sigma}. \quad (\text{V.33})$$

The dependence on ρ cancels and we are again with the full chain studied above in the Subsection V.B. Thus we have obtained the following result: there is the ρ -family of quantum algebras $U_{AE(\rho)}(g_2)$ but the full chain quantization $U_C(g_2)$ is unique.

To complete the analysis of the set of twist deformations for $U(g_2)$ let us consider the generators that can become quasiprimitive (in general nonsimultaneously) after the action of the full chain of twists \mathcal{F}_C . The number q_C^- of such generators is equal to the rank of \mathfrak{g} . In our case we have $q_C^-(g_2) = 2$ and the corresponding generators are $\widetilde{e_{0,-1}}$ and $\widetilde{e_{-1,0}}$. Both are quasiprimitive simultaneously but cannot enlarge the space of \mathfrak{g}_C^+ up to a (quasi)Frobenius subalgebra in $U(g_2)$. In the Borel subalgebra $U_C(\mathfrak{b}_+(g_2))$ the number q_C^+ of quasiprimitive generators is equal to the number of simultaneously primitive. For the full chains we have $q_C^+ = r$ and in our case (as we have seen above) these primitive generators are σ and ω . This certainly provides the possibility to perform the additional Abelian twist but the corresponding deformation will be equivalent to the redefinition of the Cartan elements in the Jordanian twisting factors.

VI. DISCUSSION AND OUTLOOK

We have described the set $\{U_J, U_{E+J}, U_{EJ}, U_{J'EJ}, U_{A(\rho)EJ}\}$ of quantized Lie-Poisson structures that were constructed on the space $U(g_2)$ by chains of twist deformations. The dual group coordinates ($\mathfrak{g}^\#$ -basis) obtained through the second classical limit procedure provide us with the possibility of writing down the explicit form of these Lie-Poisson structures. We have presented in this paper the algebraic and coalgebraic formulae determined by the exceptional Lie algebra g_2 , in particular:

- we get the additional (in comparison with the situation in $U(sl(3))$ and $U(so(5))$) Hopf algebra U_{EJ}
- the carrier space for the second Jordanian twist $\mathcal{F}_{J'}$ is deformed (the analogous result was found for $U_{E+J}(so(5))$ [11, 12])
- the Hopf algebras $\{U_{E+J}, U_{J'EJ}, U_{AEJ}\}$ in comparison with the analogous quantizations of $U(sl(3))$ and $U(so(5))$ have more complicated costructure determined by the root system of g_2 .

We have also found the peripheric twisted algebras in the set of Hopf algebras $\{U_{A(\rho)EJ}\}$. The number s_C of inequivalent algebras in this set depends on the number l_C^n of extended Jordanian factors in the chain \mathcal{F}_C . We conjecture the following relation

$$s_C = \sum_{i=1}^{l_C^n} \left(\dim(\mathfrak{g}_c^{(i)}) - 1 \right), \quad (\text{VI.34})$$

where $\mathfrak{g}_c^{(i)}$ is the carrier subalgebra of the link $\mathcal{F}^{(i)} \subset \mathcal{F}_C$. This gives three inequivalent subsets for $U_{A(\rho)EJ}(sl(3))$ as well as $U_{A(\rho)EJ}(so(5))$ and five for $U_{A(\rho)EJ}(g_2)$. One of the algebras, $U_{A(0)EJ}(g_2)$, corresponds to the canonical

extended twist and four others are the analogues of the peripheric extended deformations [12] in $U_{EJ}(sl(N))$. At the same time we do not have peripheric chains in $U_{\mathcal{F}}(g_2)$. As we have already stressed above the full chain deformation $U_{\mathcal{F}}(g_2)$ is invariant under the rotation generated by additional Abelian twist.

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